

A Class of Weighted Bergman Spaces, Reducing Subspaces for Multiple Weighted Shifts, and Dilatable Operators

by

Xiaoming Liang

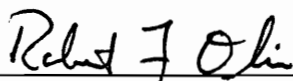
Dissertation submitted to the faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

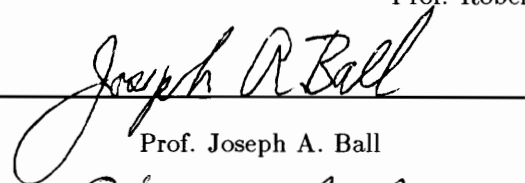
in

Mathematics

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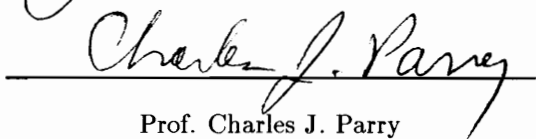
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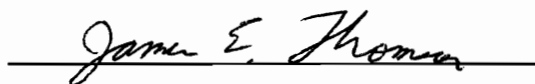
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August, 1996

Blacksburg, Virginia

Key words: Weighted Bergman Space, Weighted Shift, Reducing Subspace, Invariant Subspace, Super-Dilatable Operator

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(Abstract)

This thesis consists of four chapters. Chapter 1 contains the preliminaries. We give the background, notation and some results needed for this work, and we describe our main results of this thesis.

In Chapter 2 we will introduce a class of weighted Bergman spaces. We then will discuss some properties about the multiplication operator, M_z , on them. We also characterize the dual spaces of these weighted Bergman spaces.

In Chapter 3 we will characterize the reducing subspaces of multiple weighted shifts. The reducing subspaces of the Bergman and the Dirichlet shift of multiplicity N are portrayed from this characterization.

In Chapter 4 we will introduce the class of super-isometrically dilatable operators and describe their elementary properties. We then will discuss an equivalent description of the invariant subspace lattice for the Bergman shift. We will also discuss the interpolating sequences on the bidisk. Finally, we will examine a special class of super-isometrically dilatable operators. One corollary of this work is that we will prove that the compression of the Bergman shift on two compliments of two invariant subspaces are unitarily equivalent if and only if the two invariant subspaces are equal.

Acknowledgements

I wish to thank my advisor Robert F. Olin, for his advice, encouragement and support. I would like to thank Dr. Joseph A. Ball and Dr. Warren R. Wogen at University of North Carolina-Chapel Hill for their helpful conversations and suggestions about the results in Chapter 3 when I visited the Berkeley Mathematical Sciences Research Institute. I would like to thank Dr. James E. Thomson for his help.

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Chapter 1

INTRODUCTION

The theory about the Bergman space and the associated multiplication operator on it originated from the work of S. Bergman in 1950 [16]. The Bergman space is a typical example of a Hilbert space of analytic functions which possesses a reproducing kernel. Axler's survey paper [7] contains a very nice exposition on the theory of Bergman spaces and the natural operators defined on them. Since the proof of the universal dilation property of the Bergman shift (See [13] [14]), the Bergman shift has played a very important role in operator theory. A concrete description of its invariant subspace lattice will result in the resolution of the invariant subspace problem (the most famous open question of Functional analysis and Operator theory). The theory of weighted Bergman spaces and their associated multiplication operators has attracted the attention of many operator theorists and function theorists. Several papers that have been published have investigated other aspects of this area such as the compactness and boundedness of the Toeplitz operators and Hankel operators on the weighted Bergman space, and the algebraic and analytic relationships between the symbol and the Toeplitz operator. Many interesting results can be found in McDonald-Sundberg [70], Luecking [61] [62] [63] [64] [65], Zhu [88], Axler [5] [6] [7], Korenblum [54] [55] [56], Bonsall [19], Yu and Sun [87], Elias [32], Faour [33] [34], Faour and Yousel [35] and so on.

The structure of the lattice of invariant subspaces of M_z on $L_a^2(\mathbf{D})$ and the properties of functions in weighted Bergman spaces has been investigated in several papers. For example,

S. Richter's work [73] addresses the unitarily equivalent class of invariant subspaces. Zhu's work [90] concerns itself with similarly equivalent class of invariant subspaces. Hedenmalm's work [40] [41] [43] [44] [45], Horowitz's work [47] [48] [49], Janas's work [52], Khavinson and Shapiro's work [53] focus on the factorization of functions, zero sets of functions and the expression of invariant subspaces, etc. As spectacular and deep most of these results have been, results on the lattice of the Bergman shift have been scarce as it is known that there is no simple characterization of it like the famous theorem on the invariant subspaces of M_z on $H^2(\mathbf{D})$ due to Arne Beurling [18]. To get a glimpse why the situation of the Bergman space is so different from that of Hardy space we note that there exist in the Bergman space z -invariant subspaces I having the so-called codimension n property, for any integer $n = 1, 2, 3, \dots$; even for $n = \infty$. Here we say an invariant subspace I has the codimension n property if zI has codimension n in I . Existence of such subspaces was discovered by Constantin Apostol, Hari Bercovici, Ciprian Foias, and Carl Pearcy [14]. Using Kristian Seip's famous work in [76] [77] that gives complete description of interpolating and sampling sequences for the Bergman space $L_a^2(\mathbf{D})$ in terms of densities, H. Hedenmalm gave a constructive proof in [42] of some codimension 2 invariant subspaces. Alexandru Aleman's papers [2] [3], and Axler and Bourdon's work [8] construct the codimension n spaces. The work on factorization (see [40] [41] [45] [74]) for square area-integrable analytic function, permit the expression of invariant subspaces by using the extremal functions. However the properties of the extremal functions are scarce; hence, a penetrating characterization of the lattice of invariant subspaces of M_z on $L_a^2(\mathbf{D})$ is still not available.

Observing the history of the theory about Bergman spaces and the operators on them, we can see that most of the work have been done by using function theoretic techniques, i.e., the results have been obtained through the use of hard analysis. In this thesis, our

results care about through the use of soft analysis and operator algebra techniques. Ronald G. Douglas and V. I. Paulsen's work in [29] and Shunhua Sun and Dahai Yu's work in [83] illustrate the flavor of our work. Before we state the results of this thesis, let's introduce a basic concepts and theorems.

We will use the following notations, definitions and theorems.

- We will use $\mathcal{H}$ and $\mathcal{K}$ to denote the general separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$, $B(\mathcal{H})$ to denote the set of all bounded linear operator on $\mathcal{H}$, for an operator $T \in B(\mathcal{H})$, we will use $Lat(T)$ to denote the lattice of all invariant subspaces of T in $\mathcal{H}$, a subspace $M \subset \mathcal{H}$ is said to be an invariant subspace of T if $TM \subset M$.
- We say that an operator $T \in B(\mathcal{H})$ is normal if $TT^* = T^*T$, here T^* is the operator such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for any $x, y \in \mathcal{H}$. If $TT^* = T^*T = 1$, we say T is a unitary operator.
- An operator $S \in B(\mathcal{H})$ is subnormal if there exists a Hilbert space $\mathcal{K}$ and a normal operator $N \in B(\mathcal{K})$ such that
 - (1) $\mathcal{H} \subset \mathcal{K}$
 - (2) $S = N|_{\mathcal{H}}$
- An operator $S \in B(\mathcal{H})$ is hyponormal if $\langle Sx, Sx \rangle \geq \langle S^*x, S^*x \rangle$ for all $x \in \mathcal{H}$.
- The subspace $M \subset \mathcal{H}$ is a reducing subspace for an operator $T \in B(\mathcal{H})$ if $M \in Lat(T) \cap Lat(T^*)$.
- Let $\mathcal{B}$ be a Banach space over complex field $\mathbb{C}$, $\mathcal{B}^*$ be the set of all bounded linear transformations (i.e. linear functional) from $\mathcal{B}$ to $\mathbb{C}$, we call $\mathcal{B}^*$ the dual space of $\mathcal{B}$.

- We say that an operator $T \in B(\mathcal{H})$ is belong to c_{00} if $\lim_{n \rightarrow +\infty} \|T^n x\| = 0$ and $\lim_{n \rightarrow +\infty} \|T^{*n} x\| = 0$ for all $x \in \mathcal{H}$.
- $C_p(\mathcal{H})$ is the Schatten p-class of operators on a separable Hilbert space $\mathcal{H}$ which consists of all compact operators T with singular values $\{\lambda_n\}_{n=1}^{+\infty}$ such that

$$\|T\|_p = \left[\sum_n |\lambda_n|^p \right]^{\frac{1}{p}} < +\infty.$$

Note C_1 is the class of trace class operators. If $\{e_n\}_{n=1}^{+\infty}$ is an orthonormal basis for $\mathcal{H}$, we define

$$\text{tr}(T) = \sum_n (Te_n, e_n).$$

Let's recall some standard duality result(see [24, Page 8] or [14, Page 2]).

Proposition 1.1: *The dual space $C_1(\mathcal{H})^*$ of Banach space $C_1(\mathcal{H})$ can be identified with $B(\mathcal{H})$. This duality is implemented by the bilinear functional*

$$\langle T, K \rangle = \text{tr}(TK), \quad T \in B(\mathcal{H}), \quad K \in C_1$$

In particular, we have that

$$\|T\| = \sup\{|\langle T, K \rangle| : K \in C_1, \|K\|_1 \leq 1\}$$

and

$$\|K\|_1 = \sup\{|\langle T, K \rangle| : T \in B(\mathcal{H}), \|T\| \leq 1\}.$$

- A net $\{T_\lambda\}$ in $B(\mathcal{H})$ is *weak** convergent to an operator T_0 means for every $K \in C_1$, $\text{tr}(T_\lambda K) \rightarrow \text{tr}(T_0 K)$. This characterizes the *weak**-topology on $B(\mathcal{H})$.
- A dual algebra is a subalgebra of $B(\mathcal{H})$ that contains the identity and is closed in *weak** topology on $B(\mathcal{H})$.

- For Γ , a subset of $B(\mathcal{H})$, we call ${}^\perp\Gamma = \{K \in C_1 : \langle K, S \rangle = 0, S \in \Gamma\}$ the preannihilator of Γ .

We need the following result for our further work(see [14, Page 6] for a proof).

Proposition 1.2: *Let X be a complex Banach space and let M be a weak* closed subspace of X^* with preannihilator ${}^\perp M$. Then $X/{}^\perp M$ is a Banach space whose dual, $(X/{}^\perp M)^*$, can be identified with M . In particular, if M is a weak* closed linear manifold in $B(\mathcal{H})$, then $C_1(\mathcal{H})/{}^\perp M = Q_M$ is a Banach space whose dual space can be identified with M . Under this identification the pairing between M and Q_M is given by the bilinear functional $\langle T, [L] \rangle = \text{tr}(TL)$, $T \in M$, $[L] \in Q_M$, where, as usual, we write $[L]$ for the coset in Q_M for an element $L \in C_1(\mathcal{H})$.*

- If $x, y \in \mathcal{H}$, we denote by $x \otimes y$ the rank-one operator defined by $(x \otimes y)(u) = \langle u, y \rangle x$ for every $u \in \mathcal{H}$. Let $M \subset B(\mathcal{H})$ be a weak*-closed subspace, and let n be any cardinal number such that $1 \leq n \leq \aleph_0$. Then M will be said to have property (A_n) provided every $n \times n$ system of simultaneous equations of the form $[x_i \otimes y_j] = [L_{ij}]$, $0 \leq i, j < n$, (where the $[L_{ij}]$ are arbitrary but fixed elements from Q_M) has a solution $\{x_i\}_{0 \leq i < n}$, $\{y_i\}_{0 \leq i < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$.
- A contraction (i.e., $\|T\| \leq 1$, it is a strict contraction if strict inequality occurs) is an absolutely continuous contraction if T has no nonzero invariant subspace on which it acts as a unitary operator.
- $\mathbf{A}(\mathcal{H})$ consists of all those absolutely continuous contractions T in $B(\mathcal{H})$ for which the functional calculus $\Phi_T : H^\infty(D) \rightarrow \mathcal{A}_T$ is an isometry. (Here $\mathcal{A}_T$ is the weak* closed algebra generated by T and 1, for the definition of Φ_T see [14, Page 34]). If n is a cardinal number such that $1 \leq n \leq \aleph_0$, we denote by $\mathbf{A}_n$ the set of all T in $\mathbf{A}(\mathcal{H})$

such that the algebra $\mathcal{A}_T$ has property (A_n) .

The following theorem describes the *universal dilation property* of $\mathbf{A}_{\aleph_0}$ operators, for the proof see [13] [14, Page 48, 60].

Theorem 1.1: (H. Bercovici, B. Foias and C. Pearcy) *Suppose $T \in \mathbf{A}_{\aleph_0}$, and let $\{A_j\}_{j=0}^{\infty}$ be any sequence of strict contractions acting on Hilbert spaces of dimension less than or equal to $\aleph_0$. Then there exist two invariant subspaces M, N with $N \subset M$ for T such that $T_{M \ominus N}$ is unitarily equivalent to the direct sum $\sum_j \oplus A_j$.*

Here $T_{M \ominus N}(x) = P_{M \ominus N}T(x)$, for all $x \in M \ominus N$ and $P_{M \ominus N}$ is the orthogonal projection from the whole space to $M \ominus N$.

Proposition 1.3: *If $T \in c_{00}$ and $\sigma(T) = \mathbf{D}^-$, then $T \in \mathbf{A}_{\aleph_0}$.*

An operator A is called a unilateral weighted shift if there is an orthonormal basis $\{e_n : n \geq 0\}$ and a sequence of scalars $\{\alpha_n\}$ such that $Ae_n = \alpha_n e_{n+1}$ for all $n \geq 0$. In this case, we say that A is a weighted shift with weight sequence $\{\alpha_n\}$. When $\alpha_n = \sqrt{\frac{n+1}{n+2}}$, A is the Bergman shift; hence, by Proposition 1.3, we can see the Bergman shift belongs to $\mathbf{A}_{\aleph_0}$. Many results can be found about weighted shift in [79] [24]. The following two come from [24, Page 53] and will be used in our next chapter.

Proposition 1.4: *A weighted shift is hyponormal if and only if its weight sequence is increasing.*

Theorem 1.2: *Let $\{e_0, e_1, \dots\}$ be an orthonormal basis for $\mathcal{H}$ and let S be a weighted shift relative to this basis with weight sequence $\{\alpha_n\}$, where $\sup_n \alpha_n = 1$. The following statements are equivalent.*

(a) *S is subnormal.*

(b) *There is a probability measure ν on $[0, 1]$ containing 1 in its support such that for*

$n \geq 1$

$$(\alpha_0 \cdots \alpha_{n-1})^2 = \int r^{2n} d\nu(r).$$

(c) There is a probability measure ν on $[0, 1]$ containing 1 in its support such that if μ is the measure defined on $\mathbb{C}$ by

$$d\mu(re^{i\theta}) = (2\pi)^{-1} d\theta d\nu(r),$$

then S is unitarily equivalent to S_μ .

The correspondence between subnormal weighted shifts of norm 1 and probability measures on $[0, 1]$ with 1 in their support, as described in (b), is bijective.

Here we have denoted the operator of multiplication by z on the space $P^2(\mu) \equiv L^2(\mu) - \text{closure of polynomials}$ by the symbol S_μ .

Suppose $(\mathbf{X}, \mu)$ is a measure space and K is a function on $\mathbf{X} \times \mathbf{X}$. Let T be the integral operator induced by K ; that is,

$$Tf(x) = \int_{\mathbf{X}} K(x, y)f(y)d\mu(y).$$

Where $f \in L^p(\mathbf{X}, d\mu)$. The following theorem gives sufficient conditions for the boundedness of the operator T on $L^p(\mathbf{X}, d\mu)$ in case $1 < p < +\infty$; for the proof see [88, Page 42].

Theorem 1.3:(Schur's test) Support K is a nonnegative measurable function on $\mathbf{X} \times \mathbf{X}$, T is the integral operator induced by K , and $1 < p < +\infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If there exists a constant $C > 0$ and a positive measurable function h on $\mathbf{X}$ such that

$$\int_{\mathbf{X}} K(x, y)h(y)^q d\mu(y) \leq Ch(x)^q$$

for μ -almost every x in $\mathbf{X}$ and

$$\int_{\mathbf{X}} K(x, y)h(x)^p d\mu(x) \leq Ch(y)^p$$

for μ -almost every y in $\mathbf{X}$, then T is bounded on $L^p(\mathbf{X}, d\mu)$ with norm less than or equal to C .

In Chapter 2 we focus our attention on the operator M_z on $P^2(\mu_\alpha)$ where

$$d\mu_\alpha = \frac{1}{\Gamma(-2\alpha) \left[\ln \frac{1}{|z|^2} \right]^{2\alpha+1}} dA(z)$$

with $dA(z) = \frac{dx dy}{\pi}$. We describe when $M_z = M_{z_\alpha}$ is subnormal; we show (using these operators M_{z_α}) that the (norm) limit of a sequence of universal dilatable operators is not necessarily a universal dilatable operator; and we compute $(L_a^p(\mu_\alpha))^*$. (Here $L_a^p(\mu)$ is the space of analytic functions that are $L^p(\mu)$ integrable.) Note if

$$dA_\alpha = (-2\alpha) \frac{1}{(1 - |z|^2)^{1+2\alpha}} dA(z),$$

then some similar results can be found in [88]. Note, A_α and μ_α are mutually absolutely continuous and their Radon-Nikodym derivatives are bounded above and below. Our duality result follows from the similar result in [88]. However, our proof is independent of [88] and focuses on the reproducing kernels in $L_a^p(\mu_\alpha)$. (We want to acknowledge a helpful discussion with Alexander Aleman about this result during a visit of his to Virginia Tech.) Very little is known about the relationship between two reproducing kernels when the two corresponding measures have such a functional relationship.

In Chapter 3 we will characterize the reducing subspaces for a multiple weighted shift. As a consequence of our work, we get the characterization of the reducing subspaces of the operator M_{z^n} on $L_a^2(\mathbf{D})$; i.e., the n th power of the Bergman shift. Professors J. Ball and W. Wogen were helpful in focusing my attention on this consequence. We get similar results for the multiplication operator M_{z^n} on the Dirichlet spaces and weighted Bergman spaces. For the details, please see Chapter 3. In Chapter 4 we will introduce the class

of super-isometrically dilatable operators, first discussed in the work [83]. This is a dilation theory which is different from the one of Nagy-Foias. Sun and Yu developed some fundamental properties about this class of operators and used this theory to solve a unitarily equivalent problem about multiplication operators on Bergman space. In Chapter 4 we will develop some additional properties about this class of operators. We then use the fact that the Bergman shift is a super-isometrically dilatable operator with minimal dilation $\{M_z, M_w, H^2(D^2)\}$ to study the lattice of the Bergman shift, here $H^2(D^2)$ is the standard Hardy space on the bidisk D^2 and M_z and M_w are the multiplication operators by coordinates on $H^2(D^2)$. We will get an equivalent description of the invariant subspace problem and we also discuss the interpolating sequence problem on the bidisk. Our result about interpolating sequences uses Seip's famous work [77]. We will see the interpolating sequence problem in the bidisk is much more complicated than that in the Bergman space case. (It is easy to see that the interpolation problem in the Bergman space is a special case of the bidisk case.) Finally we will discuss a problem about the equivalent class for the lattice of the Bergman shift. For the details, please see the last section of Chapter 4.

The following are our main results in this thesis:

Theorem 2.2: (Duality of weighted Bergman space) *If $1 < p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \leq 0$, and $p(-2\alpha) > 1$, then*

$$(L_a^p(d\mu_\alpha))^* \cong L_a^q(d\mu_\alpha)$$

under the usual integral pairing

$$\langle f, g \rangle = \int_D f(z) \overline{g(z)} d\mu_\alpha(z)$$

Theorem 3.3: $\mathbf{I} \in \text{Lat}(M_{z^n}) \cap \text{Lat}(M_{z^n}^*)$ iff there exists j with $1 \leq j < n$ and there exist $i_0, i_1, \dots, i_{j-1} \in \{0, 1, \dots, n-1\}$ such that

$$\mathbf{I} = \text{Span}\{M_{z^n}^l z^{i_0}, M_{z^n}^l z^{i_1}, \dots, M_{z^n}^l z^{i_{j-1}}, 0 \leq l\}.$$

Here M_{z^n} is acting on $L_a^2(D)$.

Theorem 3.6: Let T be a weighted shift of multiplicity N with weights $\sqrt{\alpha_n}$ (see section 3.1). The following are equivalent:

(i) There exists a vector $a = (a_1, a_2, \dots, a_N)$ in the subspace

$$\text{span}\{(1, 1, \dots, 1), (\prod_{l=0}^n \alpha_{l \cdot N+1}, \prod_{l=0}^n \alpha_{l \cdot N+2}, \dots, \prod_{l=0}^n \alpha_{l \cdot N+N}) : n \geq 0\}$$

of the Euclidean space R^N such that

$$a_i \neq a_j, \quad \text{whenever } i \neq j$$

(ii) If $\mathcal{M} \in \text{Lat}(T) \cap \text{Lat}(T^*)$, and if P is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$, then P is a diagonal matrix under the base $\{e_n\}_{n=1}^\infty$ and

$$P = \text{diag}\{d_1, d_2, \dots, d_N, d_1, d_2, \dots, d_N, d_1, d_2, \dots, d_N, \dots\}$$

with $d_i = 1$ or $0, i = 1, 2, \dots, N$.

(iii) $\mathcal{M} \in \text{Lat}(T) \cap \text{Lat}(T^*)$ and $\mathcal{M} \neq \{0\}$ if and only if there exists j with $1 \leq j < N$ and there exist $i_0, i_1, \dots, i_j \in \{1, 2, \dots, N\}$ such that

$$\mathcal{M} = \mathcal{H}_{i_0} \oplus \mathcal{H}_{i_1} \oplus \dots \oplus \mathcal{H}_{i_j}$$

here $\mathcal{H}_{i_k} = \text{span}\{T^l e_{i_k}, l \geq 0\}^-$.

Lemma 3.2: *Let T be a weighted shift of multiplicity N on a Hilbert space $\mathcal{H}$, where N is finite or infinite. There exist homogeneous weighted shifts T_k for $k = 1, 2, 3, \dots, M$, where M is finite or infinite, with weight sequence $\{\sqrt{\lambda_{k,n}}\}_{n=1}^{+\infty}$ and multiplicity N_k finite or infinite, acting on Hilbert spaces $\mathcal{H}_k$ such that*

$$N = N_1 + N_2 + N_3 + \dots + N_M$$

$$T = T_1 \oplus T_2 \oplus T_3 \oplus \dots \oplus T_M$$

under the decomposition

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \dots \oplus \mathcal{H}_M$$

and for $k_i \neq k_j$ we have

$$\{\sqrt{\lambda_{k_i,n}}\}_{n=1}^{+\infty} \neq \{\sqrt{\lambda_{k_j,n}}\}_{n=1}^{+\infty}$$

Furthermore, for each k the Hilbert space $\mathcal{H}_k$ has a decomposition

$$\mathcal{H}_k = \mathcal{H}_k^{(1)} \oplus \mathcal{H}_k^{(2)} \oplus \mathcal{H}_k^{(3)} \oplus \dots \oplus \mathcal{H}_k^{(N_k)}$$

with the following properties:

- (a) *For each m , the space $\mathcal{H}_k^{(m)}$ is a reducing subspace for T_k .*
- (b) *There exists an orthonormal basis $\{e_n^{(k,m)}\}_{n=1}^{\infty}$ of $\mathcal{H}_k^{(m)}$ such that*

$$T_k e_n^{(k,m)} = \sqrt{\lambda_{k,n}} e_{n+1}^{(k,m)}$$

Note: This says T_k acting on $\mathcal{H}_k^{(m)}$ is a weighted shift and the weight sequence does not depend on m . Hence it follows that for any $x_{(n)} = \sum_{m=1}^{N_k} \beta_n^{(k,m)} e_n^{(k,m)}$, we have $T_k x_{(n)} = \sqrt{\lambda_{k,n}} x_{(n+1)}$ with $x_{(n+1)} = \sum_{m=1}^{N_k} \beta_n^{(k,m)} e_{n+1}^{(k,m)}$.

Theorem 3.9: *For the above weighted shift T on the space $\mathcal{H}$, $\mathcal{M}$ is reducing subspace for T if and only if*

$$\mathcal{M} = \sum_{k=1}^M \oplus \sum_{l=1}^{+\infty} \oplus M_k^{(l)}$$

where

$$M_k^{(l)} = \left\{ \sum_{m=1}^{N_k} \beta_1^{(k,m)} e_l^{(k,m)}; \text{ if } \sum_{m=1}^{N_k} \beta_1^{(k,m)} e_1^{(k,m)} \in M_k^{(1)} \right\}$$

and $M_k^{(1)}$ is any subspace of the space $\text{span}\{e_1^{(k,m)}; 1 \leq m < N_k + 1\}^-$, or

$$\mathcal{M} = \text{Span}\{T_1^l M_1^{(1)} \oplus T_2^l M_2^{(1)} \oplus \cdots \oplus T_M^l M_M^{(1)}; l \geq 1\}^-$$

where the spaces $M_k^{(1)}$ are the same as before.

Theorem 4.1: *Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are two invariant subspaces for T , a super isometrically dilatable operator (for the definition, please see Chapter 4) on Hilbert space $\mathcal{H}$. Also assume that the minimal super dilated triple is $\{U, V, \mathcal{K}\}$ and that the commutant $\{U, V, U^*, V^*\}'$ is trivial. Then there exists a unitary operator $\mathbf{V}$ from $\mathcal{O}_1$ onto $\mathcal{O}_2$ satisfying $\mathbf{V}S_1 = S_2\mathbf{V}$ if and only if $\mathcal{M}_1 = \mathcal{M}_2$. These conditions occur if and only if $\mathcal{O}_1 = \mathcal{O}_2$. (Here $\mathcal{O}_i = \mathcal{H} \ominus \mathcal{M}_i$ and $S_i = P_{\mathcal{O}_i}T|_{\mathcal{O}_i}$.)*

Corollary 4.2: *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be invariant subspaces for the Bergman shift on Bergman Space $L_a^2(\mathbb{D})$, Let $\mathcal{O}_i = L_a^2(\mathbb{D}) \ominus \mathcal{M}_i$, and $P_{\mathcal{O}_i}$ be the projections from $L_a^2(\mathbb{D})$ to $\mathcal{O}_i$. There exists a unitary operator $\mathbf{V}$ from $\mathcal{O}_1$ to $\mathcal{O}_2$ such that $(P_{\mathcal{O}_1}\mathcal{M}_z|_{\mathcal{O}_1})\mathbf{V} = \mathbf{V}(P_{\mathcal{O}_2}\mathcal{M}_z|_{\mathcal{O}_2})$ if and only if $\mathcal{O}_1 = \mathcal{O}_2$. Obviously these two conditions occur if and only if $\mathcal{M}_1 = \mathcal{M}_2$.*

Chapter 2

A class of weighted Bergman spaces and some basic properties

In this chapter, we will first introduce some basic terminologies and definitions regarding the weighted Bergman spaces under investigation. We then compute their dual spaces and we also study the properties of the operator multiplication by z , M_z , on them. Similar problems have been studied in many papers such as [88] and [58]. In Section 2 we will compute the dual spaces of the weighted Bergman spaces using some ideas of this latter work (but adding a new technique). In Section one, we will study the multiplication operator M_z and compute when M_z is subnormal. We will give another proof about the following result:

For the Bergman shift M , if $\alpha \geq 0$, then M^α is subnormal. In his Ph.D. dissertation [58], Li obtained the same result by using moment sequence method and Berger's theorem [58, Theorem 0.8].

2.1 Basic definitions and properties

Let D be the unit disk of complex plane, $A(D)$ be all the analytic functions on the unit disk D . For a real number α and a real number $p > 0$, D_α^p is the space defined by the following

$$D_\alpha^p = \{f \in A(D); \sum_{k=0}^{\infty} |\hat{f}(k)|^p (k+1)^{2\alpha} < +\infty\}$$

where for $k \geq 0$ we denote the Taylor coefficients of f by $\hat{f}(k) \equiv \frac{f^{(k)}(0)}{k!}$ ($k \geq 0$). We define a norm on D_α^p by

$$\|f\|_p \equiv \left(\sum_{k=0}^{\infty} |\hat{f}(k)|^p (k+1)^{2\alpha} \right)^{\frac{1}{p}}.$$

Proposition 2.1: *The space D_α^p is a Banach space under the norm $\|\cdot\|_p$, and the space D_α^2 is a Hilbert space under the following inner product*

$$\langle f(z), g(z) \rangle \equiv \sum_{k=0}^{\infty} \hat{f}(k) \overline{\hat{g}(k)} (k+1)^{2\alpha}.$$

Furthermore $\{e(z) = \frac{z^n}{(n+1)^\alpha}\}_{n=0}^{\infty}$ is an orthonormal basis for the space D_α^2 .

Proof: The proof that the space D_α^p is a normed space is a routine application of Holder's inequality. We need to prove the completeness of the space. Let $\{f_n(z)\}_{n=1}^{+\infty}$ be a Cauchy sequence in D_α^p . Since

$$\begin{aligned} & |f_n(z) - f_m(z)| \\ &= \left| \sum_{k=0}^{\infty} (\hat{f}_n(k) - \hat{f}_m(k)) z^k \right| \\ &= \left| \sum_{k=0}^{\infty} (\hat{f}_n(k) - \hat{f}_m(k)) (k+1)^{\frac{2\alpha}{p}} \frac{z^k}{(k+1)^{\frac{2\alpha}{p}}} \right| \\ & \quad (\text{By Holder's inequality}) \\ &\leq \|f_n(z) - f_m(z)\|_p \left\| \sum_{k=0}^{+\infty} \frac{|z|^{kq}}{(k+1)^{\frac{2\alpha q}{p}}} \right\|, \end{aligned}$$

we see that on any compact subset of D , the sequence $\{f_n(z)\}_{n=1}^{+\infty}$ converges uniformly to a function $f(z)$. Hence $f(z)$ is analytic on D . It's not difficult to prove $f \in D_\alpha^p$. It is also easy to check D_α^2 is a Hilbert space and that $\{e_n(z) = \frac{z^n}{(n+1)^\alpha}\}$ is an orthonormal basis.

Let $\mathcal{D}_\alpha^p$ be the Banach space of all sequences $\{c_n\}_{n=1}^{+\infty}$ satisfying

$$\sum_{k=0}^{+\infty} |c_k|^p (k+1)^{2\alpha} < +\infty$$

Now let's define a map η between this space and the space D_α^p by setting

$$\eta(\{c_k\}_{k=0}^{+\infty}) \equiv \sum_{k=0}^{+\infty} c_k z^k$$

It's not difficult to check this map is a unitary map between these two spaces. A classical result is that $(\mathcal{D}_\alpha^p)^* = \mathcal{D}_\alpha^q$ if and only if $\frac{1}{p} + \frac{1}{q} = 1$. Hence, we have

Proposition 2.2: *For the space D_α^p with $p > 1$, we have that its dual space $(D_\alpha^p)^*$ is equal to the space D_α^q if and only if $\frac{1}{p} + \frac{1}{q} = 1$.*

Now let's restrict our concentration on the Hilbert space D_α^2 and consider the multiplication operator M_z on it. It's easy to see this operator is a weighted shift with respect to the basis $e_n(z)$ in Proposition 2.1; i.e.,

$$M_z e_n(z) = \left(\frac{n+2}{n+1} \right)^\alpha e_{n+1}(z)$$

Using the Proposition 1.4, we have the following result.

Proposition 2.3: *The operator M_z on D_α^2 is hyponormal if and only if $\alpha \leq 0$. Note that when $\alpha = 0$, M_z is the unilateral shift, and when $\alpha = -\frac{1}{2}$, M_z is the Bergman shift.*

For $\alpha < 0$ let's consider the measure

$$d\mu_\alpha(re^{i\theta}) \equiv \frac{1}{\Gamma(-2\alpha)[\ln \frac{1}{|z|}]^{2\alpha+1}} dA(z)$$

on the unit disk $D = \{re^{i\theta}; 0 \leq r < 1, 0 \leq \theta < 2\pi\}$. It's not difficult to check that the space $L_a^2(d\mu_\alpha, D)$ consisting of all analytic functions f on D satisfying

$$\int_D |f(z)|^2 d\mu_\alpha(re^{i\theta}) < +\infty$$

with inner product $(f, g \in L_a^2(\mu_\alpha, D))$

$$\langle f(z), g(z) \rangle = \int_D f(z) \overline{g(z)} d\mu_\alpha(re^{i\theta})$$

is equal to the space D_α^2 , and both have the same orthogonal base stated in Proposition 2.1. Hence, the operator M_z on D_α^2 is unitarily equivalent to the operator M_z on $L_a^2(d\mu_\alpha, D)$. With this last observation we have

Proposition 2.4: *The operator M_z on D_α^2 is subnormal if and only if $\alpha \leq 0$. Note that when $\alpha = 0$, M_z is the unilateral shift, and that when $\alpha = -\frac{1}{2}$, M_z is the Bergman shift.*

Corollary 2.1: *If M is the Bergman shift; i.e., the weighted shift with weight sequence $\{\sqrt{\frac{n+1}{n+2}}\}_{n=0}^{+\infty}$ and $\alpha \geq 0$, and $\alpha \geq 0$, then the weighted shift M^α with weight sequence $\{\left(\frac{n+1}{n+2}\right)^{\alpha/2}\}_{n=0}^\infty$ is also subnormal.*

Remark 2.1: The above theorem is proved in [58] by using the Berger's Theorem [58, Theorem 0.8] and an application of the theory of moment sequences.

Remark 2.2: The kind of weighted shifts described in Corollary 2.1 belong to c_{00} class of contractions and using Theorem 6.7 in [24] we see $\sigma(M_z) = D$ for each M_z on $L_a^2(d\mu_\alpha, D)$. Hence by Proposition 1.3, we see that M_z has the universal dilation property.

Remark 2.3: Let's consider the relationship between all the operators M_z on the space $L_a^2(d\mu_\alpha(re^{i\theta}), D)$. At first let's think of each of these operators as a weighted shift M_α for any α on a Hilbert space $\mathcal{H}$ with an orthonormal basis $\{e_n\}_{n=0}^{+\infty}$; i.e.,

$$M_\alpha e_n = \left(\frac{n+2}{n+1}\right)^\alpha e_{n+1}; \quad \text{for all } n$$

Now all of these multiplication operators are acting on the same Hilbert space $\mathcal{H}$, and it is routine to show that

$$\lim_{\alpha \rightarrow 0^-} \|M_\alpha - M_0\|_{\mathcal{H}} = 0.$$

In fact,

$$(M_\alpha - M_0)e_n = \left[\left(\frac{n+2}{n+1} \right)^\alpha - 1 \right] e_{n+1}.$$

Hence, we have

$$\|M_\alpha - M_0\| = |1 - 2^\alpha|$$

Hence, M_α approaches M_0 uniformly, but their limit, M_0 , is not a universal dilatable operator.

Remark 2.4: The weighted shift $M_0 - M_\alpha$ belongs to $\mathcal{C}_p(\mathcal{H})$.

To see this, let $T = M_0 - M_\alpha$; we have

$$Te_n = \left[1 - \left(\frac{n+2}{n+1} \right)^\alpha \right] e_{n+1}$$

and an easy computation shows that

$$T^*Te_n = \left[1 - \left(\frac{n+2}{n+1} \right)^\alpha \right]^2 e_n.$$

It's not difficult to prove

$$\sum_{n=0}^{+\infty} \left[1 - \left(\frac{n+2}{n+1} \right)^\alpha \right]^p < +\infty$$

for any $\alpha < 0$ and $p > 1$. Thus we have $T \in \mathcal{C}_p(\mathcal{H})$. On the other hand, $M_\alpha + T = M_0$; i.e., for $p > 1$, we have

Proposition 2.5: *There is an operator M which has the universal dilation property and an operator T which is in $\mathcal{C}_p(\mathcal{H})$ such that the sum $M + T = M_0$, such that $M_0 \in \mathbf{A}_2$ but not in $\mathbf{A}_{\aleph_0}$*

For $p = 1$, we have a very different result:

Theorem 1: *For any hyponormal c_{00} weighted shift $M \in \mathbf{A}_{\aleph_0}(\mathcal{H})$ and for any weighted shift $T \in \mathcal{C}_1(\mathcal{H})$, if $T + M$ is in $\mathbf{A}(\mathcal{H})$, we have $T + M \in \mathbf{A}_{\aleph_0}(\mathcal{H})$.*

Note: The above theorem is a partial solution for the open problem of Hari Bercovici, Ciprian Foias and Carl Pearcy in [14]; i.e.,

Problem 10.9. If $M \in \mathbf{A}_{\mathbb{N}_0}(\mathcal{H})$ and $K \in \mathcal{C}_1(\mathcal{H})$ is such that $M + K \in \mathbf{A}(\mathcal{H})$, is it always true that $M + K \in \mathbf{A}_{\mathbb{N}_0}(\mathcal{H})$?

Proof: Suppose $Me_k = m_k e_{k+1}$ and $Te_k = t_k e_{k+1}$. Since $M \in \mathbf{A}_{\mathbb{N}_0}$, we have $T \in \mathbf{A}(\mathcal{H})$, hence, the functional calculus $\Phi_M : \mathcal{H}^\infty(D) \rightarrow \mathcal{A}_M$ is an isometry. Hence $\|M\| = 1$. By the Theorem 6.7 in [24] we have $\sigma(M) = D$. Since $T \in \mathcal{C}_1(\mathcal{H})$, hence $t_n + m_n \rightarrow \|M\| = 1$, by Proposition 6.8 in [24], we have

$$\sigma(M + T) = \{\lambda : |\lambda| \leq \|M\|\} = \sigma(M)$$

We only need to prove the sum $M + T$ is belong to c_{00} ; i.e.,

$$\lim_{l \rightarrow +\infty} \prod_{s=1}^l (t_{n+s} + m_{n+s}) = 0.$$

This is equivalent to showing that

$$\sum_{s=1}^{+\infty} (1 - t_{n+s} - m_{n+s}) = +\infty \quad (*)$$

since M is in c_{00} and $T \in \mathcal{C}_1(\mathcal{H})$, we have

$$\prod_{s=1}^{+\infty} m_{n+s} = 0 \quad i.e. \quad \sum_{s=1}^{+\infty} (1 - m_{s+n}) = +\infty$$

and

$$\sum_{s=1}^{+\infty} t_{n+s} < +\infty$$

hence $(*)$ hold, i.e $M + T$ is still in c_{00} , by proposition 1.3, we are done.

2.2 A Duality Theorem

In this last section of the chapter we define for each $\alpha < 0$ a measure μ_α on the unit disk $D = \{re^{i\theta} : r \geq 0, 0 \leq \theta < 2\pi\}$ by setting

$$d\mu_\alpha(re^{i\theta}) \equiv \frac{1}{\Gamma(-2\alpha)[\ln \frac{1}{|z|^2}]^{2\alpha+1}} dA(z).$$

Now let's consider the space $L_a^p(d\mu_\alpha(re^{i\theta}), D)$ consisting of all analytic function f on D satisfying

$$\int_D |f(z)|^p d\mu_\alpha(re^{i\theta}) < +\infty$$

a natural question comes up

Do we still have the Duality theorem? That is, is the following result true ?

$$(L_a^p(d\mu_\alpha, D))^* = L_a^q(d\mu_\alpha, D)$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

The answer for the last question is affirmative. Before we prove this theorem, let's set up some basic properties about these weighted Bergman spaces. We have the following lemma about the reproducing kernels on these spaces.

Lemma 2.1: *If $\alpha \geq 0$, there exists a number C_α such that*

$$\frac{(n+1)^\alpha}{(1+\frac{\alpha}{n})(1+\frac{\alpha}{n-1}) \cdots (1+\alpha)} \leq C_\alpha$$

for all n .

Proof: It's easy to prove when $\alpha = k$ a nonnegative integer that

$$\lim_{n \rightarrow +\infty} \frac{(n+1)^\alpha}{(1+\frac{\alpha}{n})(1+\frac{\alpha}{n-1}) \cdots (1+\alpha)} = k!.$$

On the other hand, for each n , let

$$f_n(\alpha) = \frac{(n+1)^\alpha}{(1+\frac{\alpha}{n})(1+\frac{\alpha}{n-1})\cdots(1+\alpha)}.$$

One shows that

$$\frac{df_n}{d\alpha}(\alpha) = f_n(\alpha) \left(\ln(n+1) - \frac{1}{n+\alpha} - \frac{1}{(n-1)+\alpha} - \cdots - \frac{1}{1+\alpha} \right).$$

The last result implies that the function $f_n(\alpha)$ is increasing; hence, the value of $f_n(\alpha)$ for non integer α must be between $f_n(k)$ and $f_n(k+1)$. We easily see our conclusion.

Lemma 2.2: *Let*

$$K_\alpha(z, w) = \sum_{n=0}^{+\infty} e_n(z) \overline{e_n(w)} = \sum_{n=0}^{+\infty} \frac{z^n \overline{w}^n}{(n+1)^{2\alpha}}.$$

For each $f \in L_a^2(d\mu_\alpha, D)$

$$f(z) = \int_D f(w) K_\alpha(z, w) d\mu_\alpha(w).$$

Here $\{e_n(z)\}_{n=0}^{+\infty}$ is any orthogonal basis of the space $L_a^2(d\mu_\alpha, D)$.

The proof of the above lemma is similar to the proof of the similar result about standard Bergman space, we omit it.

Now let's consider the projection P_α defined by

$$P_\alpha f(z) = \int_D K_\alpha(z, w) f(w) d\mu(w).$$

It's easy to see $P_\alpha f = f$ for all analytic f in $L_a^1(d\mu_\alpha, D)$ (just consider $f(rz)$ and take the appropriate limit). Furthermore we have

Lemma 2.3: For $\alpha < 0$, the projection P_α defined above is bounded from $L^p(d\mu_\alpha, D)$ to $L^p_a(d\mu_\alpha, D)$ and onto, for $p > 1$.

Proof: We will use Schur's theorem to prove this lemma. Let's consider the function

$$h(z) = \frac{1}{(1 - |z|^2)^{\frac{1}{pq}}}$$

where q is a positive real number such that $\frac{1}{p} + \frac{1}{q} = 1$. According to the Schur's theorem, our result follows if we establish the following two inequalities:

$$\int_D |K_\alpha(z, w)| h(w)^q d\mu_\alpha(w) \leq C h(z)^q$$

and

$$\int_D |K_\alpha(z, w)| h(z)^p d\mu_\alpha(z) \leq C h(w)^p$$

for some constant C . Since

$$d\mu_\alpha(z) = \frac{1}{\Gamma(-2\alpha)} \frac{1}{\left[\ln \left(\frac{1}{|z|^2} \right) \right]^{1+2\alpha}} dA(z)$$

we have

$$\begin{aligned} & \int_D h(z)^p |K_\alpha(z, w)| d\mu_\alpha(z) \\ &= \int_D \frac{1}{(1 - |z|^2)^{\frac{1}{q}}} \left| \sum_{m=0}^{+\infty} \frac{z^m \bar{w}^m}{(m+1)^{2\alpha}} \right| \frac{1}{\Gamma(-2\alpha)} \frac{1}{\left[\ln \left(\frac{1}{|z|^2} \right) \right]^{1+2\alpha}} dA(z) \end{aligned}$$

Interchanging the sum and the integral via Fubini theorem, our last equality is equal to

$$\sum_{m=0}^{+\infty} \int_D \frac{1}{(1 - |z|^2)^{\frac{1}{q}}} \frac{|z|^m |w|^m}{(m+1)^{2\alpha}} \frac{1}{\Gamma(-2\alpha)} \frac{1}{\left[\ln \left(\frac{1}{|z|^2} \right) \right]^{1+2\alpha}} dA(z).$$

Since

$$\frac{1}{\Gamma(-2\alpha)} \frac{1}{\left[\ln \left(\frac{1}{|z|^2} \right) \right]^{1+2\alpha}}.$$

is comparable to $\frac{1}{(1-|z|^2)^{1+2\alpha}}$ when $|z| \rightarrow 1^-$, there is a constant C_α such that

$$\begin{aligned} & \int_D h(z)^p |K_\alpha(z, w)| d\mu_\alpha(z) \\ & \leq C_\alpha \sum_{m=0}^{+\infty} \int_D \frac{|z|^m |w|^m}{(m+1)^{2\alpha} (1-|z|^2)^{\frac{1}{q}+1+2\alpha}} dA(z). \end{aligned}$$

Recall the inequality in 4.2.3 Theorem [88], we have

$$\int_D (1+\beta) \left| \frac{(1-|z|^2)^\beta}{(1-w\bar{z})^{2+\beta}} \right| h(z)^p dA(z) \leq Ch(w)^p.$$

For $\beta > -1$ and $p(\beta+1) > 1$ let $\beta = -1 - 2\alpha$ and note

$$\frac{1}{(1-z\bar{w})^{1-2\alpha}} = \sum_{n=0}^{+\infty} \frac{\Gamma(n+1-2\alpha)}{n!\Gamma(1-2\alpha)} z^n \bar{w}^n.$$

Hence,

$$\sum_{m=0}^{+\infty} \int_D \frac{\Gamma(m+1-2\alpha)}{m!\Gamma(1-2\alpha)} \frac{|z|^m |w|^m}{(1-|z|^2)^{\frac{1}{q}+1+2\alpha}} dA(z) \leq Ch(z)^p$$

We now compare the following two series;

$$\begin{aligned} & \sum_{n=0}^{+\infty} \frac{|z|^n |w|^n}{(n+1)^{2\alpha}}, \\ & \sum_{n=0}^{+\infty} \frac{\Gamma(n+1-2\alpha)}{n!\Gamma(1-2\alpha)} |z|^n |w|^n. \end{aligned}$$

At first, we can easily see that

$$\begin{aligned} & \frac{\Gamma(n+1-2\alpha)}{n!\Gamma(1-2\alpha)} \\ & = \frac{(1-2\alpha+n-1)(1-2\alpha+n-2)\cdots(1-2\alpha)\Gamma(1-2\alpha)}{n!\Gamma(1-2\alpha)} \\ & = \frac{(n-2\alpha)(n-1-2\alpha)\cdots(1-2\alpha)}{n!} \\ & = \left(1 - \frac{2\alpha}{n}\right) \left(1 - \frac{2\alpha}{n-1}\right) \cdots (1-2\alpha) \end{aligned}$$

Using lemma 2.1, we have

$$\frac{(n+1)^{-2\alpha}}{(1 - \frac{2\alpha}{n})(1 - \frac{2\alpha}{n-1}) \cdots (1 - 2\alpha)} \leq C_\alpha$$

for some constant C_α and all n , now our conclusion follows.

Theorem 2.2: (Duality of weighted Bergman space) *If $1 < p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and if $\alpha \leq 0$, and if $p(-2\alpha) > 1$, then*

$$(L_a^p(d\mu_\alpha))^* \cong L_a^q(d\mu_\alpha)$$

under the usual integral pairing

$$\langle f, g \rangle = \int_D f(z) \overline{g(z)} d\mu_\alpha(z)$$

Proof: This follows from the Hahn-Banach extension theorem and the boundedness of the projection P_α on $L_a^p(d\mu_\alpha, D)$.

Chapter 3

On Reducing Subspaces of Multiple Weighted Shift

3.1 Introduction

Let $\mathbf{D}$ be the open unit disk in the complex plane $\mathbf{C}$. The Bergman space, $L_a^2(\mathbf{D})$ is the closure in $L^2(\mathbf{D})$ of the analytic polynomials. The space, $L_a^\infty(\mathbf{D})$ is the weak-star closure in $L^\infty(\mathbf{D})$ of polynomials. For a function $\phi \in L_a^\infty(\mathbf{D})$, the multiplication operator, M_ϕ , is defined on $L_a^2(\mathbf{D})$ as follows:

$$(M_\phi f)(z) = \phi(z)f(z) \quad \text{for any } f \in L_a^2(\mathbf{D}).$$

When $\phi(z) = z$, we call M_ϕ the Bergman shift, and when $\phi(z) = z^N$, we call M_ϕ the Bergman shift of multiplicity N .

We know for the Bergman shift M_z , its commutant is $L_a^\infty(\mathbf{D})$ (see [24, Page 73, Theorem 8.17]). So it is easy to prove that M_z has no nontrivial reducing subspace, but for $N > 1$, of course M_ϕ has a nontrivial reducing subspace, but what do they look like? For any given invariant subspace for M_ϕ , can we say if it is a reducing subspace or not? In this chapter we will give the characterization of reducing subspaces of M_{z^N} on $L_a^2(\mathbf{D})$. When $N = 1$ our result will give the algebraic proof: the Bergman shift has no nontrivial reducing subspace.

At first let's consider the general case. Let $\mathcal{H}$ be a separable Hilbert space, l^∞ be the Banach space of all bounded sequences, $\{\sqrt{\alpha_n}\}$ be a bounded positive sequence; i.e.,

there exists a positive number C such that $0 < \sqrt{\alpha_n} < C$. Now let T be the weighted shift of multiplicity $N < \infty$ under the orthonormal base e_n , $n = 1, 2, \dots$ of $\mathcal{H}$ with weight $\sqrt{\alpha_n}$; i.e.,

$$Te_n = \sqrt{\alpha_n}e_{n+N}$$

$n = 1, 2, \dots$. We want to emphasize that the only criterion we need right now on the weights for the operator T , is that they are bounded and positive. Their actual value are irrelevant at this moment. We are interested in the lattice of reducing subspaces for T . For convenience, let

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots \mathcal{H}_N,$$

Where $\mathcal{H}_k = \text{Span}\{T^l e_k; l \geq 0\}^-$, $k = 1, 2, \dots, N$.

3.2 Theorems and their proofs.

The theorems in this section are not our main theorems of this chapter, but they lead to our main theorems in section 4.

Theorem 3.1: *Assume for the weighted shift of multiplicity N above, that the N vectors of the following*

$$(1, \alpha_k, \alpha_k \alpha_{N+k}, \alpha_k \alpha_{N+k} \alpha_{2N+k}, \dots, \prod_{l=0}^{n-1} \alpha_{l \cdot N+k}, \dots)$$

$k = 1, 2, \dots, N$ are linear independent in l^∞ . We have the following;

(a) *If $\mathcal{M} \in \text{Lat}(T) \cap \text{Lat}(T^*)$ and if P is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$, then P is a diagonal matrix under the base $\{e_n\}_{n=0}^\infty$ and*

$$P = \text{diag}\{d_1, d_2, \dots, d_N, d_1, d_2, \dots, d_N, \dots, d_1, d_2, \dots, d_N, \dots\}$$

with $d_i = 1$ or $0, i = 1, 2, \dots, N$.

(b) $\mathcal{M} \in \text{Lat}(T) \cap \text{Lat}(T^*)$ and $\mathcal{M} \neq \{0\}$ if and only if there exists j with $1 \leq j \leq N$ and there exist $i_0, i_1, \dots, i_{j-1} \in \{1, 2, \dots, N\}$ such that

$$\mathcal{M} = \mathcal{H}_{i_0} \oplus \mathcal{H}_{i_1} \oplus \dots \oplus \mathcal{H}_{i_{j-1}},$$

where $\mathcal{H}_{i_k} = \text{span}\{T^l e_{i_k}, l \geq 0\}^-$.

Proof of Theorem 3.1: Since $\mathcal{M} \in \text{Lat}(T) \cap \text{Lat}(T^*)$, the operator T can be written as $A \oplus B$, with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. Hence $\ker T^* = \ker A^* \oplus \ker B^*$. Since P is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$. We have $A = PTP = TP$. Note

$$\begin{aligned} \ker T^* &= \text{Span}\{e_1, e_2, \dots, e_N\}^- \\ &= \ker A^* \oplus \ker B^*. \end{aligned}$$

Since the sequence $\{\sqrt{\alpha_n}\}_{n=1}^{+\infty}$ is bounded and positive and both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant for T . we have

$$\begin{aligned} \mathcal{H} &= \text{Span}\{T^l(\ker T^*); l \geq 0\}^- \\ &= \text{Span}\{T^l(\ker A^*); l \geq 0\}^- \oplus \text{Span}\{T^l(\ker B^*); l \geq 0\}^-. \end{aligned}$$

Thus, we only need prove that

$$\ker A^* = \text{Span}\{e_{i_0}, e_{i_1}, \dots, e_{i_{j-1}}\}^-$$

for some subset $\{i_0, i_1, \dots, i_{j-1}\}$ of $\{1, 2, \dots, N\}$. Note $j = \dim(\ker A^*)$. Now let's observe

$$\ker A^* = \text{Span}\{Pe_1, Pe_2, \dots, Pe_N\}^-$$

and

$$\ker B^* = \text{Span}\{(1-P)e_1, (1-P)e_2, \dots, (1-P)e_N\}^-.$$

Thus, for any fixed k with $1 \leq k \leq N$, we only need to prove $Pe_k = e_k$ or 0 . Since

$$T^*Pe_k = PT^*e_k = 0,$$

and

$$T^*(1 - P)e_k = 0,$$

we can suppose

$$Pe_k = y_{k,1}e_1 + y_{k,2}e_2 + \cdots + y_{k,k}e_k + \cdots + y_{k,N}e_N$$

and

$$(1 - P)e_k = f_{k,1}e_1 + f_{k,2}e_2 + \cdots + f_{k,k}e_k + \cdots + f_{k,N}e_N.$$

Hence,

$$e_k = \sum_{i=1}^N (y_{k,i} + f_{k,i})e_i.$$

This implies that $y_{k,k} + f_{k,k} = 1$ and $y_{k,i} + f_{k,i} = 0$ for $k \neq i$. Hence,

$$(1 - P)e_k = -y_{k,1}e_1 - \cdots + (1 - y_{k,k})e_k - \cdots - y_{k,N}e_N.$$

For any $l \geq 0$, the vector $T^l Pe_k$ is perpendicular to $T^l(1 - P)e_k$. Hence,

$$\langle T^l Pe_k, T^l(1 - P)e_k \rangle = 0.$$

We now see that

$$\sum_{i=1}^N |y_{k,i}|^2 = y_{k,k},$$

and

$$\sum_{i=1}^N \left(\prod_{j=0}^{l-1} \alpha_{jN+i} \right) |y_{k,i}|^2 = y_{k,k} \left(\prod_{j=0}^{l-1} \alpha_{jN+k} \right)$$

for $l = 1, 2, \dots$. Obviously, $y_{k,k}$ is a real number. If $y_{k,k} = 0$, we have $y_{k,i} = 0$ for any i .

Hence, we have $Pe_k = 0$. If $y_{k,k} \neq 0$, let $x_i = \frac{|y_{k,i}|^2}{y_{k,k}}$. We have

$$\sum_{i=1}^N x_i = 1$$

and

$$\sum_{i=1}^N \left(\prod_{j=0}^{l-1} \alpha_{jN+i} \right) x_i = \left(\prod_{j=0}^{l-1} \alpha_{jN+k} \right)$$

for $l = 1, 2, \dots$. By assumption the N vectors

$$(1, \alpha_j, \alpha_j \alpha_{N+j}, \alpha_j \alpha_{N+j} \alpha_{2N+j}, \dots, \prod_{l=0}^{n-1} \alpha_{l \cdot N + j}, \dots)$$

$j = 1, 2, \dots, N-1$ are linear independent; hence, the above linear system has at most one solution. It is easy to see there is a solution which is $x_1 = 0, \dots, x_k = 1, \dots, x_N = 0$. Hence, we have $Pe_k = e_k$, thus (a) is proved.

(b) follows from (a) easily.

Theorem 3.2: Let $\{\lambda_n\}$ be a positive sequence such that $\sqrt{\frac{\lambda_n}{\lambda_{n+1}}}$ bounded and bounded below. Let $Te_n = \sqrt{\frac{\lambda_n}{\lambda_{n+1}}} e_{n+1}$. For each $l = 1, 2, \dots, N$ let

$$\mathcal{H}_l = \text{Span}\{T^{Nm} e_l; m \geq 0\}^\perp$$

Assume there exist integers $k_1, k_2, \dots, k_N \geq 0$ such that

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ \frac{\lambda_1}{\lambda_{k_1 N + 1}} & \frac{\lambda_2}{\lambda_{k_1 N + 2}} & \frac{\lambda_3}{\lambda_{k_1 N + 3}} & \dots & \frac{\lambda_N}{\lambda_{k_1 N + N}} \\ \frac{\lambda_1}{\lambda_{k_2 N + 1}} & \frac{\lambda_2}{\lambda_{k_2 N + 2}} & \frac{\lambda_3}{\lambda_{k_2 N + 3}} & \dots & \frac{\lambda_N}{\lambda_{k_2 N + N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda_1}{\lambda_{k_{N-1} N + 1}} & \frac{\lambda_2}{\lambda_{k_{N-1} N + 2}} & \frac{\lambda_3}{\lambda_{k_{N-1} N + 3}} & \dots & \frac{\lambda_N}{\lambda_{k_{N-1} N + N}} \end{vmatrix} \neq 0$$

or,

$$\begin{vmatrix} \frac{1}{\lambda_{k_1 N+1}} & \frac{1}{\lambda_{k_1 N+2}} & \frac{1}{\lambda_{k_1 N+3}} & \cdots & \frac{1}{\lambda_{k_1 N+N}} \\ \frac{1}{\lambda_{k_2 N+1}} & \frac{1}{\lambda_{k_2 N+2}} & \frac{1}{\lambda_{k_2 N+3}} & \cdots & \frac{1}{\lambda_{k_2 N+N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\lambda_{k_N N+1}} & \frac{1}{\lambda_{k_N N+2}} & \frac{1}{\lambda_{k_N N+3}} & \cdots & \frac{1}{\lambda_{k_N N+N}} \end{vmatrix} \neq 0$$

For the P and $\mathcal{M}$ we have the same conclusion as in Theorem 3.1. That is,

(a) If $\mathcal{M} \in \text{Lat}(T^N) \cap \text{Lat}((T^N)^*)$, P is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$, then P is a diagonal matrix under the base $\{e_n\}_{n=1}^\infty$ and

$$P = \text{diag}\{d_1, d_2, \dots, d_N, d_1, d_2, \dots, d_N, \dots, d_1, d_2, \dots, d_N, \dots\}$$

with $d_i = 1$ or $0, i = 1, 2, \dots, N$.

(b) $\mathcal{M} \in \text{Lat}(T^N) \cap \text{Lat}((T^N)^*)$ and $\mathcal{M} \neq \{0\}$ if and only if there exists j with $1 \leq j < N$ and there exist $i_0, i_1, \dots, i_{j-1} \in \{1, 2, \dots, N\}$ such that

$$\mathcal{M} = \mathcal{H}_{i_0} \oplus \mathcal{H}_{i_1} \oplus \cdots \oplus \mathcal{H}_{i_{j-1}}$$

here $\mathcal{H}_{i_k} = \text{span}\{T^l e_{i_k}, l \geq 0\}^-$.

Proof of Theorem 3.2 For the operator T^N , the corresponding N vectors are

$$(1, \frac{\lambda_j}{\lambda_{N+j}}, \frac{\lambda_j}{\lambda_{2N+j}}, \dots, \frac{\lambda_j}{\lambda_{nN+j}}, \dots)$$

$j = 1, 2, \dots, N$. The hypothesis about the determinants force the above N vectors to be linear independent in l^∞ . Hence, we have the conclusions of this theorem by using Theorem 3.1.

Now let's focus our attention on the Bergman shift of multiplicity N , M_{z^N} , First we need the following result on Vandermond determinant.

Lemma 3.1: *Let x_i and y_j be any variables, where $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$. We have the following:*

$$\begin{vmatrix} (x_1 - y_2)(x_1 - y_3) \cdots (x_1 - y_n) & (x_1 - y_1)(x_1 - y_3) \cdots (x_1 - y_n) & \cdots & (x_1 - y_1)(x_1 - y_2) \cdots (x_1 - y_{n-1}) \\ (x_2 - y_2)(x_2 - y_3) \cdots (x_2 - y_n) & (x_2 - y_1)(x_2 - y_3) \cdots (x_2 - y_n) & \cdots & (x_2 - y_1)(x_2 - y_2) \cdots (x_2 - y_{n-1}) \\ (x_3 - y_2)(x_3 - y_3) \cdots (x_3 - y_n) & (x_3 - y_1)(x_3 - y_3) \cdots (x_3 - y_n) & \cdots & (x_3 - y_1)(x_3 - y_2) \cdots (x_3 - y_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ (x_n - y_2)(x_n - y_3) \cdots (x_n - y_n) & (x_n - y_1)(x_n - y_3) \cdots (x_n - y_n) & \cdots & (x_n - y_1)(x_n - y_2) \cdots (x_n - y_{n-1}) \end{vmatrix} \\ = (-1)^{C_n^2} \xi^{1/2}(x_1, x_2, \dots, x_n) \cdot \xi^{1/2}(y_1, y_2, \dots, y_n)$$

Here $\xi^{1/2}(x_1, x_2, \dots, x_n)$ and $\xi^{1/2}(y_1, y_2, \dots, y_n)$ are the Vandermond determinants determined by $x_1, x_2, \dots, x_n$, and $y_1, y_2, \dots, y_n$ respectively and $C_n^2 = \frac{n!}{2!(n-2)!}$.

Proof: Let D_n be the value of the determinant above. If $x_i = x_j$ for some $i \neq j$, then $D_n = 0$, hence D_n is divisible by $x_i - x_j$. Note also, by definition,

$$\xi^{1/2}(x_1, x_2, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j);$$

hence, D_n is divisible by $\xi^{1/2}(x_1, x_2, \dots, x_n)$. Similarly D_n is divisible by $\xi^{1/2}(y_1, y_2, \dots, y_n)$.

Computing (as a polynomial) the degree of D_n and the degree of

$$\xi^{1/2}(x_1, x_2, \dots, x_n) \xi^{1/2}(y_1, y_2, \dots, y_n),$$

we can easily see they are equal. Hence, there is a constant C such that

$$D_n = C \cdot \xi^{1/2}(x_1, x_2, \dots, x_n) \xi^{1/2}(y_1, y_2, \dots, y_n).$$

Now let $x_1 = y_1, x_2 = y_2, \dots, x_n = y_n$. In this case

$$\begin{aligned}
D_n &= \begin{vmatrix} (y_1 - y_2)(y_1 - y_3) \cdots (y_1 - y_n) & 0 & \cdots & 0 \\ 0 & (y_2 - y_1)(y_2 - y_3) \cdots (y_2 - y_n) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (y_n - y_1)(y_n - y_2) \cdots (y_n - y_{n-1}) \end{vmatrix} \\
&= (-1)^{C_n^2} \xi(y_1, y_2, \dots, y_n)
\end{aligned}$$

Hence, we have

$$D_n = (-1)^{C_n^2} \xi^{1/2}(x_1, x_2, \dots, x_n) \xi^{1/2}(y_1, y_2, \dots, y_n).$$

Theorem 3.3: For the Bergman shift of multiplicity N , M_{z^N} , $\mathbf{I} \in \text{Lat}(M_{z^N}) \cap \text{Lat}(M_{z^N}^*)$ iff there exists j with $1 \leq j < N$ and there exist $i_0, i_1, \dots, i_{j-1} \in \{0, 1, \dots, N-1\}$ such that

$$\mathbf{I} = \text{Span}\{M_{z^N}^l z^{i_0}, M_{z^N}^l z^{i_1}, \dots, M_{z^N}^l z^{i_{j-1}}, 0 \leq l\}.$$

Proof of Theorem 3.3: In Theorem 3.2, let $\lambda_n = n$ for $n = 1, 2, 3, \dots$. The Bergman shift M_z is unitarily equivalent to the weighted shift

$$M_z e_n(z) = \sqrt{\frac{n}{n+1}} e_{n+1}(z)$$

Note $e_n(z) = \sqrt{n} z^{n-1}$, $n = 1, 2, \dots$ is the orthogonal base for the Bergman space $L_a^2(\mathbf{D})$.

The appropriate determinant for the operator $M_{z^N} = (M_z)^N$ (to apply Theorem 3.2 to it) is

$$\text{Det}\left(\frac{i+1}{iN+i+1}\right)_{N \times N}.$$

Its value is

$$= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{N+1} & \frac{2}{N+2} & \frac{3}{N+3} & \cdots & \frac{N}{N+N} \\ \frac{1}{2N+1} & \frac{2}{2N+2} & \frac{3}{2N+3} & \cdots & \frac{N}{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(N-1)N+1} & \frac{2}{(N-1)N+2} & \frac{3}{(N-1)N+3} & \cdots & \frac{N}{(N-1)N+N} \end{vmatrix}$$

Subtracting the first column from every column, we have

$$Det\left(\frac{i+1}{iN+i+1}\right)_{N \times N} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{N+1} & \frac{1 \cdot 1 \cdot N}{(N+2)(N+1)} & \frac{2 \cdot 1 \cdot N}{(N+3)(N+1)} & \cdots & \frac{(N-1) \cdot 1 \cdot N}{2N(N+1)} \\ \frac{1}{2N+1} & \frac{1 \cdot 2 \cdot N}{(2N+2)(2N+1)} & \frac{2 \cdot 2 \cdot N}{(2N+3)(2N+1)} & \cdots & \frac{(N-1) \cdot 2 \cdot N}{3N(2N+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(N-1)N+1} & \frac{1 \cdot (N-1) \cdot N}{((N-1)N+2)((N-1)N+1)} & \frac{2 \cdot (N-1) \cdot N}{((N-1)N+3)((N-1)N+1)} & \cdots & \frac{(N-1) \cdot (N-1) \cdot N}{N^2((N-1)N+1)} \end{vmatrix}$$

$$= N^{N-1}[(N-1)!]^2 \left(\prod_{k=1}^{N-1} \frac{1}{kN+1} \right) \begin{vmatrix} \frac{1}{N+2} & \frac{1}{N+3} & \cdots & \frac{1}{2N} \\ \frac{1}{2N+2} & \frac{1}{2N+3} & \cdots & \frac{1}{3N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(N-1)N+2} & \frac{1}{(N-1)N+3} & \cdots & \frac{1}{N^2} \end{vmatrix}_{(N-1) \times (N-1)}$$

and by the last lemma, we have the last quantity

$$= (-1)^{C_{N-1}^2} N^{N-1} [(N-1)!]^2 \xi^{1/2}(N, 2N, \dots, (N-1)N) \xi^{1/2}(-2, -3, \dots, -N) \prod_{l=1}^{N-1} \prod_{i=1}^N \frac{1}{lN+i} \neq 0$$

(Where $\xi^{1/2}(x_1, x_2, \dots, x_{N-1}) \neq 0$ iff $x_i \neq x_j$, when $i \neq j$), and since

$$\ker M_{z^N}^* = \text{Span}\{1, z, z^2, \dots, z^{N-1}\},$$

We are done!

3.3 Other Spaces

Let $A(\mathbf{D})$ be all of the analytic functions on the unit disk $\mathbf{D}$, and α be a real number.

Recall the following space

$$D_\alpha^2 = \{f \in A(\mathbf{D}); \sum_{k=0}^{\infty} |\hat{f}(k)|^2 (k+1)^{2\alpha} < \infty\}$$

where $\hat{f}(k) = \frac{f^{(k)}(0)}{k!}$, ($k \geq 0$), the Taylor coefficient of f , equipped with the inner product

$$(f, g) = \sum_{k=0}^{\infty} \hat{f}(k) \cdot \overline{\hat{g}(k)} \cdot (k+1)^{2\alpha}$$

is a Hilbert space. We easily see that for $\alpha = 0$, this space is the classical Hardy space. For $\alpha = -\frac{1}{2}$, this space is the Bergman space; and for $\alpha = \frac{1}{2}$, this space is the Dirichlet space. Furthermore a routine calculation shows that the space D_α^2 has an orthogonal base $\{\epsilon_k(z) = \frac{z^k}{(k+1)^\alpha}\}_{k=0}^\infty$. We are interested when the operator M_{z^N} on these spaces D_α^2 have the same reducing subspaces as the Bergman shift, We just check the appropriate determinant like that one in the proof of theorem 3.2. At first let's consider the corresponding system of equations:

$$\sum_{i=0}^{n-1} \frac{(ln+i+1)^{2\alpha} |\alpha_i|^2}{(i+1)^{2\alpha}} = \alpha_k \frac{(ln+k+1)^{2\alpha}}{(k+1)^{2\alpha}} \quad l = 0, 1, 2, \dots$$

It is natural to evaluate the associated determinant:

$$D_n(\alpha) \equiv Det\left(\frac{(ln+i+1)^{2\alpha}}{(i+1)^{2\alpha}}\right)_{n \times n}$$

$$= \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ \left(\frac{n+1}{1}\right)^{2\alpha} & \left(\frac{n+2}{2}\right)^{2\alpha} & \left(\frac{n+3}{3}\right)^{2\alpha} & \dots & \left(\frac{n+n}{n}\right)^{2\alpha} \\ \left(\frac{2n+1}{1}\right)^{2\alpha} & \left(\frac{2n+2}{2}\right)^{2\alpha} & \left(\frac{2n+3}{3}\right)^{2\alpha} & \dots & \left(\frac{3n}{n}\right)^{2\alpha} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(\frac{(n-1)n+1}{1}\right)^{2\alpha} & \left(\frac{(n-1)n+2}{2}\right)^{2\alpha} & \left(\frac{(n-1)n+3}{3}\right)^{2\alpha} & \dots & \left(\frac{(n-1)n+n}{n}\right)^{2\alpha} \end{vmatrix}$$

Note in the Hardy space ($\alpha = 0$), we have $D_n(0) = 0$ if $n > 1$. At this time let us note that for the shift M_{z^n} on the Hardy space there are other kinds of reducing subspaces than described previously. For example, the following invariant space

$$Span\{M_{z^n}^l p(z); l \geq 0\}$$

Where $p(z)$ is a polynomial with $\deg(p(z)) < n$. For the Dirichlet Space let $\alpha = \frac{1}{2}$, we have

$$D_n\left(\frac{1}{2}\right) = \text{Det}\left(\frac{ln+i+1}{i+1}\right)_{n \times n}$$

$$= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{n+1}{1} & \frac{n+2}{2} & \frac{n+3}{3} & \cdots & \frac{n+n}{n} \\ \frac{2n+1}{1} & \frac{2n+2}{2} & \frac{2n+3}{3} & \cdots & \frac{3n}{n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(n-1)n+1}{1} & \frac{(n-1)n+2}{2} & \frac{(n-1)n+3}{3} & \cdots & \frac{(n-1)n+n}{n} \end{vmatrix}$$

Subtracting column i from $(i-1)$ for $i = 2, \dots, n$, we see

$$D_n\left(\frac{1}{2}\right) = \begin{vmatrix} \frac{-n}{2 \cdot 1} & \frac{-n}{3 \cdot 2} & \cdots & \frac{-n}{n(n-1)} \\ \frac{-2n}{2 \cdot 1} & \frac{-2n}{3 \cdot 2} & \cdots & \frac{-2n}{n(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-(n-1)n}{2 \cdot 1} & \frac{-(n-1)n}{3 \cdot 2} & \cdots & \frac{-(n-1)n}{n(n-1)} \end{vmatrix}_{(n-1) \times (n-1)} = 0$$

From this result, we see that when $n > 2$, for the multiplication operator M_{z^n} on the Dirichlet space, our method of the proof for the structure of the reducing subspaces is not valid. We are interested in the range of α when $D(\alpha) \neq 0$ for $n \geq 2$. Before we continue discuss this, let's introduce a new concept about orthogonal projection on these kinds of spaces.

Definition: Let $\mathbf{I}$ be a subspace of D_α^2 , and let P be the orthogonal projection from D_α^2 onto $\mathbf{I}$. If for each integer $k \geq 0$, we have $Pz^k = z^k$ or $Pz^k = 0$, we then say the projection

is a nice projection and $\mathbf{I}$ a nice subspace.

Now we have the following theorem:

Theorem 3.4: *Consider the Space D_α^2 and the multiplication operator M_{z^n} . For any $\mathbf{I} \in \text{Lat}(M_{z^n}) \cap \text{Lat}(M_{z^n}^*)$, the following are equivalent:*

- (a) $\mathbf{I}$ is a nice subspace;
- (b) The projection P from D_α^2 onto $\mathbf{I}$ is a nice projection;
- (c) There is $0 \leq j < n$ and $i_0, i_1, \dots, i_j \in \{0, 1, \dots, n-1\}$ such that

$$\mathbf{I} = \text{Span}\{M_{z^n}^l z^{i_0}, M_{z^n}^l z^{i_1}, \dots, M_{z^n}^l z^{i_j}, 0 \leq l\}^-;$$

Proof: By our definition, (a) and (b) are the same.

If (b) is satisfied, since $\mathbf{I}$ is a reducing subspace of M_{z^n} a similar proof like that in Theorem 3.2 shows (c).

If (c) is satisfied, for any $k \geq 0$, there are two nonnegative integers l and s such that $k = ln + s$. We can easily prove that if s is not in $\{i_0, i_1, \dots, i_j\}$, z^k is perpendicular to space $\mathbf{I}$ and if s is in $\{i_0, i_1, \dots, i_j\}$, $z_k \in \mathbf{I}$. Hence, we have $Pz^k = 0$ or $Pz^k = z^k$; i.e., (b) is satisfied. We are done.

Theorem 3.5: *For the Space D_α^2 and the multiplication operator M_{z^n} , if the determinant $D_n(\alpha) \neq 0$, then every $\mathbf{I} \in \text{Lat}(M_{z^n}) \cap \text{Lat}(M_{z^n}^*)$ is a nice subspace.*

Proof: The same as Theorem 3.2.

Corollary 3.1: *For the Hilbert space D_α^2 if $\alpha \neq 0$ the operator M_{z^2} has only two nontrivial reducing subspaces; they are $\text{Span}\{M_{z^2}^l \cdot 1; l \geq 0\}^-$ and $\text{Span}\{M_{z^2}^l z; l \geq 0\}^-$.*

Remark 3.1: It is natural to ask if the condition $D \neq 0$ in Theorem 3.2 is necessary.

The following example shows that it is not.

Consider the Dirichlet space and the shift M_{z^3} on it. According to our previous computation, we have $D(\frac{1}{2}) = 0$. But at this time, we still have the same conclusion as theorem 3.1.

To see this consider the following three vectors:

$$(1, 4, 7, \dots, 3(n-1) + 1, \dots)$$

$$(1, 5/2, 8/2, \dots, (3(n-1) + 2)/2, \dots)$$

$$(1, 6/3, 9/3, \dots, (3(n-1) + 3)/3, \dots)$$

It's easy to prove the above three vectors are linear dependent. Hence all of the determinants like specified in theorem 3.2 are zero. Now let P be the projection such that $PM_{z^3} = M_{z^3}P$. Suppose $Pe_0 = \alpha_0e_0 + \alpha_1e_1 + \alpha_2e_2$. Using our previous argument, we have the following system of equations:

$$|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 = \alpha_0$$

$$4|\alpha_0|^2 + \frac{5}{2}|\alpha_1|^2 + \frac{6}{3}|\alpha_2|^2 = 4\alpha_0.$$

Multiplying the first equation by 4 and subtracting the second, we get

$$\frac{3}{2}|\alpha_1|^2 + \frac{6}{3}|\alpha_2|^2 = 0.$$

This forces $\alpha_1 = 0$ and $\alpha_2 = 0$. Hence $\alpha_0 = 0$ or 1; i.e., $Pe_0 = e_0$ or 0. Similarly we can prove $Pe_2 = e_2$ or 0. Now suppose $Pe_1 = \alpha_0e_0 + \alpha_1e_1 + \alpha_2e_2$. If $Pe_0 = 0$ and $Pe_2 = 0$, then the equality $P^2 = P$ implies $Pe_1 = e_1$ or 0. Similarly for the case $Pe_0 = e_0$, $Pe_2 = 0$ and the other two cases. Hence for each $i = 0, 1, 2$, $Pe_i = e_i$ or 0. The conclusion of Theorem 3.2 is still true.

Remark 3.2: Our Theorem 3.1 and its proof are also valid for the case $N = +\infty$, To see this first note the weighted shift of multiplicity infinity can be written the following form:

$$T = T_1 \oplus T_2 \oplus \cdots \oplus T_k \oplus \cdots$$

under the decomposition

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_k \oplus \cdots$$

Here $\mathcal{H}_k$ is a separable Hilbert space with orthogonal base $\{e_{k,n}\}_{n=1}^{+\infty}$ and T_k is a weighted shift of multiplicity one with weight $\{\sqrt{\lambda_{k,n}}\}_{n=1}^{+\infty}$. That is,

$$T_k e_{k,n} = \sqrt{\lambda_{k,n}} e_{k,n+1}.$$

If the following set of vectors

$$(1, \lambda_{j,1}, \lambda_{j,1}\lambda_{j,2}, \lambda_{j,1}\lambda_{j,2}\lambda_{j,3}, \cdots, \prod_{k=1}^n \lambda_{j,k}, \cdots)$$

$j = 1, 2, \cdots$ is linearly independent set in the space l^∞ , and if P is a orthogonal projection such that $PT = TP$, then P is a diagonal matrix with the following form

$$P = P_* \oplus P_* \oplus \cdots$$

Here $P_* = \text{diag}(d_1, d_2, \cdots, d_n, \cdots)$ using the basis $e_{k,1}, e_{k,2}, \cdots, e_{k,n}, \cdots$. It is easy to see the other conclusions of Theorem 3.1 are valid.

3.4 The construction of reducing subspaces

From section 3.2, we know that the determinant condition stated in Theorem 3.2 let us obtain the complete description of the reducing subspace of the Bergman shift of multiplicity N . However, we have described the lattice of reducing subspaces for the Dirichlet

shift of multiplicity 3, even though the corresponding determinants are zero. In this section we work towards a characterization of the lattice of reducing subspaces for a general weighted shift of multiplicity N . Consider U , the unilateral shift of multiplicity N . That is,

$$\mathcal{K} = \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \cdots.$$

Where $\mathcal{H}$ a Hilbert space and $N = \dim(\mathcal{H})$ and for any $x = (x_1, x_2, x_3, \cdots) \in \mathcal{K}$ we have

$$U(x) = U(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots).$$

In [39] Paul Halmos showed that every reducing subspace has the form

$$\mathcal{M} = M \oplus M \oplus M \oplus M \cdots$$

for any M , a subspace of $\mathcal{H}$. (We will call a reducing subspace of this form a **Halmos style** reducing subspace.)

We now turn our attention to a characterization of reducing subspaces for a general weighted shift of multiplicity N . We note that Alan Lambert in [57] obtained a description of this lattice (description is in operator algebra terminology for an operator-valued weighted shift). In [60] Fang Lu gave a characterization of this lattice for a weighted shift of multiplicity N where the sequence of weights is eventually constant. The following theorem gives a new description.

Theorem 3.6: *For the weighted shift of multiplicity N described in section 3.1, the following are equivalent:*

(i) *There exists a vector $a = (a_1, a_2, \cdots, a_N)$ in the subspace*

$$\text{span}\{(1, 1, \cdots, 1), (\prod_{l=0}^n \alpha_{l \cdot N+1}, \prod_{l=0}^n \alpha_{l \cdot N+2}, \cdots, \prod_{l=0}^n \alpha_{l \cdot N+N}) : n \geq 0\}$$

of the Euclidean space R^N such that

$$a_i \neq a_j, \quad \text{whenever } i \neq j$$

(ii) If $\mathcal{M} \in \text{Lat}(T) \cap \text{Lat}(T^*)$, P is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$, then P is a diagonal matrix under the base $\{e_n\}_{n=1}^\infty$ and

$$P = \text{diag}\{d_1, d_2, \dots, d_N, d_1, d_2, \dots, d_N, \dots, d_1, d_2, \dots, d_N, \dots\}$$

with $d_i = 1$ or $0, i = 1, 2, \dots, N$.

(iii) $\mathcal{M} \in \text{Lat}(T) \cap \text{Lat}(T^*)$ and $\mathcal{M} \neq \{0\}$ if and only if there exists j with $1 \leq j < N$ and there exist $i_0, i_1, \dots, i_j \in \{1, 2, \dots, N\}$ such that

$$\mathcal{M} = \mathcal{H}_{i_0} \oplus \mathcal{H}_{i_1} \oplus \dots \oplus \mathcal{H}_{i_j}$$

here $\mathcal{H}_{i_k} = \text{span}\{T^l e_{i_k}, l \geq 0\}^-$.

Proof: We first establish that condition (i) implies (ii). As in the proof of theorem 3.1, let $\mathcal{M} \in \text{Lat}(T) \cap \text{Lat}(T^*)$. The operator T can be written as $A \oplus B$, with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. Hence $\ker T^* = \ker A^* \oplus \ker B^*$. Since P is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$, we have $A = PTP = TP$. Note

$$\begin{aligned} \ker T^* &= \text{Span}\{e_1, e_2, \dots, e_N\}^- \\ &= \ker A^* \oplus \ker B^* \end{aligned}$$

Since the sequence $\{\sqrt{\alpha_n}\}_{n=1}^{+\infty}$ is bounded and positive and both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant for T , we have

$$\begin{aligned} \mathcal{H} &= \text{Span}\{T^l(\ker T^*); l \geq 0\}^- \\ &= \text{Span}\{T^l(\ker A^*); l \geq 0\}^- \oplus \text{Span}\{T^l(\ker B^*); l \geq 0\}^- \end{aligned}$$

Thus, we only need to prove that

$$\ker A^* = \text{Span}\{e_{i_0}, e_{i_1}, \dots, e_{i_{j-1}}\}^-,$$

where $j = \dim(\ker A^*)$ and $i_0, i_1, \dots, i_{j-1} \in \{1, 2, \dots, N\}$. Since

$$\ker A^* = \text{Span}\{Pe_1, Pe_2, \dots, Pe_N\}^-$$

and

$$\ker B^* = \text{Span}\{(1 - P)e_1, (1 - P)e_2, \dots, (1 - P)e_N\}^-,$$

for any fixed k with $1 \leq k \leq N$, we only need to prove $Pe_k = e_k$ or 0 . Since

$$T^*Pe_k = PT^*e_k = 0$$

and

$$T^*(1 - P)e_k = 0,$$

we can suppose

$$Pe_k = y_{k,1}e_1 + y_{k,2}e_2 + \dots + y_{k,k}e_k + \dots + y_{k,N}e_N$$

and

$$(1 - P)e_k = f_{k,1}e_1 + f_{k,2}e_2 + \dots + f_{k,k}e_k + \dots + f_{k,N}e_N.$$

Hence,

$$e_k = \sum_{i=1}^N (y_{k,i} + f_{k,i})e_i.$$

This implies $y_{k,k} + f_{k,k} = 1$ and $y_{k,i} + f_{k,i} = 0$ for $k \neq i$. Hence

$$(1 - P)e_k = -y_{k,1}e_1 - \dots + (1 - y_{k,k})e_k - \dots - y_{k,N}e_N$$

and for any $l \geq 0$, T^lPe_k is perpendicular to $T^l(1 - P)e_k$. Since

$$\langle T^lPe_k, T^l(1 - P)e_k \rangle = 0$$

we have

$$\sum_{i=1}^N |y_{k,i}|^2 = y_{k,k}$$

and

$$\sum_{i=1}^N \left(\prod_{j=0}^{l-1} \alpha_{jN+i} \right) |y_{k,i}|^2 = y_{k,k} \left(\prod_{j=0}^{l-1} \alpha_{jN+k} \right)$$

for $l = 1, 2, 3, \dots$.

Since there exists a vector $a = (a_1, a_2, \dots, a_N)$ in the space

$$\text{span}\left\{ (1, 1, \dots, 1), \left(\prod_{l=0}^n \alpha_{l \cdot N+1}, \prod_{l=0}^n \alpha_{l \cdot N+2}, \dots, \prod_{l=0}^n \alpha_{l \cdot N+N} \right) : n \geq 0 \right\}$$

with $a_i \neq a_j$ whenever $i \neq j$, and because all of these numbers are real, we can suppose $a_1 > a_2 > a_3 > \dots > a_N$. Now we have

$$\sum_{i=1}^N |y_{k,i}|^2 = y_{k,k} \quad (1)$$

and

$$\sum_{i=1}^N a_i |y_{k,i}|^2 = a_k y_{k,k} \quad (2)$$

Now for $k = 1$, let's prove $Pe_1 = e_1$ or 0. That is, we will prove $y_{1,1} = 0$ or 1 and $y_{1,i} = 0$ when $i \neq 1$. Multiplying equation (1) by a_1 and subtracting equation (2), we have

$$\sum_{i=2}^N (a_1 - a_i) |y_{1,i}|^2 = 0 \quad (3)$$

Since $a_1 > a_i$ (hence $a_1 - a_i > 0$), we obtain $y_{1,i} = 0$ for $i = 2, 3, \dots, N$. Using equation (1) again we have $y_{1,1} = 1$ or $y_{1,1} = 0$. That is, we have $Pe_1 = e_1$ or 0. Since P is self-adjoint, we $y_{i,1} = 0$ for $i = 2, 3, \dots, N$. Now for $k = 2$, we have

$$\sum_{i=2}^N |y_{2,i}|^2 = y_{2,2} \quad (1)$$

and

$$\sum_{i=2}^N a_i |y_{2,i}|^2 = a_2 y_{2,2} \quad (2)$$

Noticing that a_2 is the largest in the set $\{a_2, a_3, \dots, a_N\}$, we use a similar argument to obtain $y_{2,2} = 1$ or 0 and $y_{2,i} = 0, y_{i,2} = 0$ for $i \neq 2$. This method of proof is clear. Hence we have $y_{i,i} = 1$ or 0 and $y_{i,j} = 0$ for $i \neq j$, so (ii) is true.

(ii) implies (iii) is obvious.

(iii) implies (i):

We will use method by contradiction to do this. At first let's prove the following claim,

Claim: *If there exists a vector $b = (b_1, b_2, \dots, b_N)$ in the space*

$$\text{span}\{(1, 1, \dots, 1), (\prod_{l=0}^n \alpha_{l \cdot N+1}, \prod_{l=0}^n \alpha_{l \cdot N+2}, \dots, \prod_{l=0}^n \alpha_{l \cdot N+N}) : n \geq 0\}$$

such that $b_i \neq b_l$ for any $i, l \geq 1$ and $i \neq l$ then condition (i) is satisfied.

To see this, let $c_1^{(0)} = (c_{1,1}, c_{1,2}, \dots, c_{1,N})$ be a vector with $c_{1,1} \neq c_{1,2}$. Let $c_{1,k}$ be the last index such that $c_{1,i} \neq c_{1,l}$ whenever $i \neq l$ and $i, l \leq k$. We know $2 \leq k$. If $k = N$, we are done. If not, we have $c_{1,k+1} = c_{1,j}$ for some $j \leq k$. Now choose a vector

$$b_{k+1}^{(0)} = (b_{k+1,1}, \dots, b_{k+1,j}, \dots, b_{k+1,k+1}, \dots, b_{k+1,N})$$

where $b_{k+1,j} \neq b_{k+1,k+1}$. Let

$$c_1^{(1)} = c_1 + \delta \cdot b_{k+1}^{(0)}$$

where δ is a number such that $|\delta \cdot b_{k+1,l}| < \frac{1}{3}d$ for $l \leq k+1$, and $d = \min\{d_1, d_2\}$ where

$$d_1 = \min\{|c_{1,i} - c_{1,l}| : i \neq l, i, l \leq k\},$$

$$d_2 = \min\{|c_{1,i} - c_{1,k+1}| : i \neq j, i \leq k\}.$$

At this time for any $i, l \leq k+1$, with $i \neq l$ and $(i, l) \neq (j, k+1)$, we have

$$\begin{aligned}
& |c_{1,i}^{(1)} - c_{1,l}^{(1)}| \\
&= |c_{1,i} + \delta b_{k+1,i} - c_{1,l} - \delta b_{k+1,l}| \\
&\geq |c_{1,i} - c_{1,l}| - \frac{2}{3}d \\
&\geq d - \frac{2}{3}d = \frac{1}{3}d > 0.
\end{aligned}$$

We have for $(i, l) = (j, k+1)$

$$\begin{aligned}
& |c_{1,j}^{(1)} - c_{1,k+1}^{(1)}| \\
&= |c_{1,j} + \delta b_{k+1,j} - c_{1,k+1} - \delta b_{k+1,k+1}| \\
&= \delta |b_{k+1,j} - b_{k+1,k+1}| > 0.
\end{aligned}$$

Using induction and this last argument, we may obtain a vector c_1^* such that all of its coordinates are distinct from each other. Hence condition (i) is satisfied.

Now we turn to the proof that (iii) $\Rightarrow$ (i). Suppose condition (i) is false. From the claim, we may suppose for convenience the first two coordinates are equal for any vector in that space. We will construct an reducing subspace which is not of the form stated in (iii). In fact, let

$$M = \text{span}\{T^l(\frac{1}{2}(e_1 + e_2)) : l \geq 0\}^-.$$

Since for any n , $\prod_{l=0}^n \alpha_{l \cdot N+1} = \prod_{l=0}^n \alpha_{l \cdot N+2}$, we have $\alpha_{l \cdot N+1} = \alpha_{l \cdot N+2}$ for any $l \geq 0$. Hence $\sqrt{\alpha_{l \cdot N+1}} = \sqrt{\alpha_{l \cdot N+2}}$ for any $l \geq 0$. Thus, we have

$$T(e_{l \cdot N+1} + e_{l \cdot N+2}) = \sqrt{\alpha_{(l+1)N+1}}(e_{(l+1)N+1} + e_{(l+1)N+2})$$

for $l = 0, 1, 2, \dots$. It's not difficult to show M is a reducing subspace of T , but it's not a type in (iii). This is a contradiction. Hence we are done.

Remark 3.3: The theorem 3.6 and its proof are also valid for the case $N = +\infty$, let T and $\mathcal{H}$ be the same as in Remark 3.2. hence the condition (i) becomes

(i'): There exists a vector $a = (a_1, a_2, \dots, a_n, \dots)$ in the space

$$\text{span}\{(1, 1, \dots, 1, \dots), (\prod_{k=1}^n \lambda_{1,k}, \prod_{k=1}^n \lambda_{2,k}, \dots, \prod_{k=1}^n \lambda_{m,k}, \dots) : n \geq 1\}$$

in the Banach space l^∞ such that

$$a_i \neq a_j, \quad \text{whenever } i \neq j.$$

For the proof of the corresponding claim, just be careful to choose the δ such that the necessary sequence $c_1^{(n)}$ converges.

Remark 3.4: It's not hard to see our determinant condition and linear independent condition are special cases of condition (i). So the Theorems 3.1 and 3.2 are special cases of Theorem 3.6.

As a consequence we have

Theorem 3.7: For the Space D_α^2 with $\alpha \neq 0$ and the multiplication operator M_{z^n} , $\mathbf{I} \in \text{Lat}(M_{z^n}) \cap \text{Lat}(M_{z^n}^*)$ iff There exist $0 \leq j < n$ and $i_0, i_1, \dots, i_j \in \{0, 1, \dots, n-1\}$ such that

$$\mathbf{I} = \text{Span}\{M_{z^n}^l z^{i_0}, M_{z^n}^l z^{i_1}, \dots, M_{z^n}^l z^{i_j}, 0 \leq l\}^-;$$

(When $\alpha = \frac{1}{2}$, M_z is the Dirichlet shift and when $\alpha = -\frac{1}{2}$, M_z is the Bergman shift.)

Proof: From the definition of M_z on D_α^2 (see chapter 2), we can see the condition (i) is satisfied.

Remark 3.5: In the last part of the proof in theorem 3.6, the choice of the vector $\frac{1}{2}(e_1 + e_2)$ can be any vector in the space $\text{span}\{e_1, e_2\}$, hence for any subspace in

$N \in \text{span}\{e_1, e_2\}$ the space $\text{span}\{T^l N; l \geq 0\}$ is reducing subspace of T .

Now it's time to characterize the reducing subspace for general weighted N shift T , in order to do this, let's generalize the Halmos's result about the unilateral N shift we mentioned before. The following result will be our another building block for describing the reducing subspaces.

Theorem 3.8: *Let T be a weighted shift of multiplicity N as defined in Remark 3.2. Suppose $\{\sqrt{\lambda_{k,n}}\}_{n=1}^\infty = \{\sqrt{\lambda_{i,n}}\}_{n=1}^\infty$ for all i and k . We have $M \in \text{Lat}(T) \cap \text{Lat}(T^*)$ iff M is Halmos style reducing subspace.*

Proof: Let I be the $N \times N$ identity matrix and $H_{(n)} = \text{Span}\{e_{k,n}, k \geq 1\}^\perp$. Hence, under the decomposition

$$\mathcal{H} = H_{(1)} \oplus H_{(2)} \oplus H_{(3)} \oplus \cdots,$$

T has a matrix form

$$\begin{pmatrix} 0 & \lambda_{1,1}I & 0 & 0 & \cdots \\ 0 & 0 & \lambda_{1,2}I & 0 & \cdots \\ 0 & 0 & 0 & \lambda_{1,3}I & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now for any projection P , with matrix form under the same decomposition

$$\begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots \\ P_{21} & P_{22} & P_{23} & \cdots \\ P_{31} & P_{32} & P_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

by using a routine matrix computation, we have that $TP = PT$ and $P = P^*$ iff

$$P = P_{11} \oplus P_{22} \oplus P_{33} \oplus \cdots$$

and all P_{nn} (for $n \geq 1$) are the same. Hence, we are done.

Remark 3.6: For convenience, let's call an operator in Theorem 3.8 a **Homogeneous Weighted Shift** with multiplicity N . When $N = 1$, it becomes our ordinary weighted shift. Of course we allow N to be infinite.

Now we have the following lemma:

Lemma 3.2: *Let T be a weighted shift of multiplicity N , where N may be infinite. There exist homogeneous weighted shifts T_k for $k = 1, 2, 3, \dots, M$ where M can be infinite with corresponding weight sequences $\{\sqrt{\lambda_{k,n}}\}_{n=1}^{+\infty}$ and corresponding multiplicity N_k finite or infinite such that*

$$N = N_1 + N_2 + N_3 + \dots + N_M$$

$$T = T_1 \oplus T_2 \oplus T_3 \oplus \dots \oplus T_M$$

under the decomposition

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \dots \oplus \mathcal{H}_M$$

and for $k \neq i$ the weight sequence $\{\sqrt{\lambda_{k,n}}\}_{n=1}^{+\infty}$ does not equal the weight sequence $\{\sqrt{\lambda_{i,n}}\}_{n=1}^{+\infty}$. Furthermore, we may express each $\mathcal{H}_k$

$$\mathcal{H}_k = \mathcal{H}_k^{(1)} \oplus \mathcal{H}_k^{(2)} \oplus \mathcal{H}_k^{(3)} \oplus \dots \oplus \mathcal{H}_k^{(N_k)}$$

where $\mathcal{H}_k^{(m)} = \text{span}\{e_n^{(k,m)}; n \geq 1\}^-$ and $T_k e_n^{(k,m)} = \sqrt{\lambda_{k,n}} e_{n+1}^{(k,m)}$ is independent on m . Hence, we have that for any $x_{(n)} = \sum_{m=1}^{N_k} \beta_n^{(k,m)} e_n^{(k,m)}$, we have $T_k x_{(n)} = \sqrt{\lambda_{k,n}} x_{(n+1)}$ with $x_{(n+1)} = \sum_{m=1}^{N_k} \beta_n^{(k,m)} e_{n+1}^{(k,m)}$.

Proof: Suppose $T = S_1 \oplus S_2 \oplus \dots \oplus S_N$ under the decomposition $\mathcal{H} = H_1 \oplus H_2 \oplus \dots \oplus H_N$, where each S_k is a weighted shift with multiplicity one on H_k and

$$S_k e_{k,n} = \sqrt{s_{k,n}} e_{k,n+1}$$

where $\{e_{k,n}\}_{n=1}^{+\infty}$ is an orthonormal base for H_k . Now if S_k and $S_{k'}$ have the same weight sequence, we collect both of them together as one homogeneous weighted shift acting on the sum of their Hilbert spaces H_k and $H_{k'}$. It's easy to prove that our decomposition is well defined and the multiplicity of each homogeneous weighted shift is the number of S_k which has been collected together in the homogeneous weighted shift. It's easy to see our classification satisfy all the other conclusions of the lemma.

For the rest of this chapter we only study the direct sum of homogeneous weighted shifts. According to the last lemma, this is equivalent to the study of general multiple weighted shifts. Now fix the weighted shift

$$T = T_1 \oplus T_2 \oplus T_3 \oplus \cdots \oplus T_M$$

acting on the Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \cdots \oplus \mathcal{H}_M$$

where each T_k is a homogeneous weighted shift. The following is our second main result of this chapter.

Theorem 3.9: *Fix a weighted shift T acting on a Hilbert space $\mathcal{H}$ as stated in the preceding paragraph. $\mathcal{M}$ is reducing subspace for T if and only if $\mathcal{M}$ satisfies one of the following two conditions:*

(i)

$$\mathcal{M} = \sum_{k=1}^M \oplus \sum_{l=1}^{+\infty} \oplus M_k^{(l)}$$

where

$$M_k^{(l)} = \left\{ \sum_{m=1}^{N_k} \beta_1^{(k,m)} e_i^{(k,m)}; \text{ if } \sum_{m=1}^{N_k} \beta_1^{(k,m)} e_1^{(k,m)} \in M_k^{(1)} \right\}$$

and $M_k^{(1)}$ is any subspace of the space $\text{span}\{e_1^{(k,m)}; 1 \leq m \leq N_k\}^-$

(ii)

$$\mathcal{M} = \text{Span}\{T_1^l M_1^{(1)} \oplus T_2^l M_2^{(1)} \oplus \cdots \oplus T_M^l M_M^{(1)}; l \geq 0\}^-$$

where the $M_k^{(1)}$ is a subspace as stated in (i) for $1 \leq k \leq M$.

Proof: The sufficient part is obvious. We prove the necessity. As in our previous proofs we compute P , the projection from $\mathcal{H}$ onto $\mathcal{M}$. From the definition of T , we know

$$\ker T^* = \text{Span}\{e_1^{(k,m)}; k, m \geq 1\}^-.$$

Since $\mathcal{M}$ is reducing subspace for T (Thus, $PT^* = T^*P$, which in turn implies $\ker T^*$ is invariant for P), and since

$$\mathcal{M} = \text{Span}\{T^l(P\ker T^*); l \geq 0\}^-.$$

Hence we need only to compute $P|_{\ker T^*}$. Note

$$\begin{aligned} \ker T^* &= \ker T_1^* \oplus \ker T_2^* \oplus \cdots \oplus \ker T_M^* \\ &= \text{span}\{e_1^{(1,m)}; m \geq 1\}^- \oplus \text{span}\{e_1^{(2,m)}; m \geq 1\}^- \oplus \cdots \oplus \text{span}\{e_1^{(M,m)}; m \geq 1\}^- . \end{aligned}$$

We prove $P\ker T_k^* \subset \ker T_k^*$, for $k \geq 1$ at first. For any $x_{(1)}^{(k)} \in \ker T_k^*$ with

$$x_{(1)}^{(k)} = \sum_{m=1}^{N_k} \beta_1^{(k,m)} e_1^{(k,m)}$$

we have

$$\begin{aligned} T^l x_{(1)}^{(k)} &= T_k^l x_{(1)}^{(k)} \\ &= T_k^{l-1} \sqrt{\lambda_{k,1}} x_{(2)}^{(k)} \\ &= \sqrt{\lambda_{k,1} \lambda_{k,2} \cdots \lambda_{k,l}} x_{(l+1)}^{(k)} \end{aligned}$$

where $x_{(l)}^{(k)} = \sum_{m=1}^{N_k} \beta_1^{(k,m)} e_l^{(k,m)}$ for $l \geq 1$. Suppose

$$Px_{(1)}^{(k)} = y_{(1)}^{(1)} \oplus y_{(1)}^{(2)} \oplus \cdots \oplus y_{(1)}^{(M)}.$$

We have

$$(I - P)x_{(1)}^{(k)} = (-y_{(1)}^{(1)}) \oplus (-y_{(1)}^{(2)}) \oplus \cdots \oplus (-y_{(1)}^{(k-1)}) \oplus (-y_{(1)}^{(k+1)}) \oplus \cdots \oplus (-y_{(1)}^{(M)}) \oplus (x_{(1)}^{(k)} - y_{(1)}^{(k)})$$

Noting that

$$\langle T^l Px_{(1)}^{(k)}, T^l(I - P)x_{(1)}^{(k)} \rangle = 0$$

for $l \geq 0$, we have

$$\begin{aligned} \sum_{m=1}^M \|y_{(1)}^{(m)}\|^2 &= \langle y_{(1)}^{(k)}, x_{(1)}^{(k)} \rangle \\ \sum_{m=1}^M \left(\prod_{n=1}^l \lambda_{m,n} \right) \|y_{(l+1)}^{(m)}\|^2 &= \left(\prod_{n=1}^l \lambda_{k,n} \right) \langle y_{(l+1)}^{(k)}, x_{(l+1)}^{(k)} \rangle. \end{aligned}$$

For $l \geq 1$, it's not difficult to see $\langle y_{(l+1)}^{(k)}, x_{(l+1)}^{(k)} \rangle = \langle y_{(1)}^{(k)}, x_{(1)}^{(k)} \rangle$ and $\|y_{(l+1)}^{(m)}\|^2 = \|y_{(1)}^{(m)}\|^2$.

For $l \geq 0$, by the construction of T we see the space

$$\text{span}\left\{ (1, 1, \dots, 1, \dots), \left(\prod_{k=1}^n \lambda_{1,k}, \prod_{k=1}^n \lambda_{2,k}, \dots, \prod_{k=1}^n \lambda_{m,k}, \dots \right) : n \geq 1 \right\}$$

satisfies the condition of the claim in Theorem 3.6. Hence, we have a vector $(a_1, a_2, \dots, a_M)$ in this space such that $a_i \neq a_j$ whenever $i \neq j$. For convenience, let's suppose $a_1 > a_2 > a_3 > \dots$. Hence, we have

$$\begin{aligned} \sum_{m=1}^M \|y_{(1)}^{(m)}\|^2 &= \langle y_{(1)}^{(k)}, x_{(1)}^{(k)} \rangle \\ \sum_{m=1}^M a_m \|y_{(1)}^{(m)}\|^2 &= a_k \langle y_{(1)}^{(k)}, x_{(1)}^{(k)} \rangle. \end{aligned}$$

Thus for $k = 1$ we have $y_1^{(m)} = 0$ for $m \geq 2$. That is, $Px_{(1)}^{(1)} \in \ker T_1^*$. Since $x_{(1)}^{(1)}$ is arbitrary, we have $P(\ker T_1^*) \subset \ker T_1^*$ and since $P^* = P$ we have

$$P(\ker T_k^*) \subset \ker T_2^* \oplus \ker T_3^* \oplus \cdots \oplus \ker T_M^*.$$

For $k \geq 2$, and for any $x_{(1)}^{(2)}$, we have the corresponding $y_{(1)}^{(1)} = 0$. Hence, we have

$$\begin{aligned}\sum_{m=2}^M \|y_{(1)}^{(m)}\|^2 &= \langle y_{(1)}^{(2)}, x_{(1)}^{(2)} \rangle \\ \sum_{m=2}^M a_m \|y_{(1)}^{(m)}\|^2 &= a_2 \langle y_{(1)}^{(2)}, x_{(1)}^{(2)} \rangle.\end{aligned}$$

Since $a_2 > a_3 > \dots$, we have $y_{(1)}^{(m)} = 0$ for $m \geq 3$. That is, $P(\ker T_2^*) \subset \ker T_2^*$. Again noting $P^* = P$, we have

$$P(\ker T_k^*) \subset \oplus \ker T_3^* \oplus \ker T_4^* \oplus \dots \oplus \ker T_M^*.$$

For $k \geq 3$, repeat this argument again to obtain

$$P(\ker T_k^*) \subset \ker T_k^*, \quad \text{for any } k \geq 1.$$

Hence,

$$P|_{\ker T^*} = P_1 \oplus P_2 \oplus \dots \oplus P_M$$

and each P_k is an orthogonal projection on $\ker T_k^*$. We are done.

Chapter 4

Super-Isometrically Dilatable operators and Bergman Shift

4.1 Introduction

Let $\mathcal{K}$ and $\mathcal{H}$ be Hilbert Spaces, and as before, $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ denote the sets of all bounded linear operators on $\mathcal{H}$ and $\mathcal{K}$, respectively. A pair of operators U and V in $\mathcal{B}(\mathcal{K})$ is doubly commuting if $UV = VU$ and $UV^* = V^*U$. A bounded linear operator T in $\mathcal{B}(\mathcal{H})$ is super-isometrically dilatable if there exists a pair of doubly commuting isometries U and V on a Hilbert Space $\mathcal{K} \supset \mathcal{H}$ such that

$$i) \quad P_{\mathcal{H}}U^iV^j|_{\mathcal{H}} = T^{i+j}, \quad \text{for any } i, j \geq 0$$

and

$$ii) \quad U^*|_{\mathcal{H}} = V^*|_{\mathcal{H}} = T^*.$$

Here $P_{\mathcal{H}}$ is the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$. The triple $\{U, V, \mathcal{K}\}$ is said to be a super-isometric dilation of T . The triple $\{U, V, \mathcal{K}\}$ is said to be minimal if $\mathcal{K}$ is the minimal common reducing subspace for U and V that contains $\mathcal{H}$. More details about super isometrically dilatable operators can be found in [83]. Let $L_a^2(D)$ denote the standard Bergman space of all analytic functions on the open unit disk D in the complex plane $\mathbb{C}$ that satisfy the integrability condition

$$\|f\|_{L^2} = \left(\int_D |f(z)|^2 dA(z) \right)^{\frac{1}{2}} < +\infty$$

Here A denotes area measure in the complex plane $\mathbf{C}$, normalized by a constant factor:

$$dA(z) = dxdy/\pi, \quad z = x + iy.$$

Let $\mathcal{H}^2(T^2)$ be the Hardy space on the polydisk D^2 with the Haar measure σ on the torus T^2 , the distinguished boundary of the polydisk D^2 . For any $\phi \in L^\infty(D)$, the Toeplitz operator T_ϕ on $L_a^2(D)$ will be defined as following:

$$T_\phi f = P(\phi f), \quad \text{for any } f \in L_a^2(D).$$

Here P is the projection from $L^2(D)$ to $L_a^2(D)$. For any $\psi \in L^\infty(T^2)$, the Toeplitz operator T_ψ on $H^2(T^2)$ will be defined similarly. That is,

$$T_\psi f = P(\psi f), \quad \text{for any } f \in H^2(T^2).$$

Again P denote the projection from $L^2(T^2)$ to $H^2(T^2)$.

As we said before, the structure of the lattice of invariant subspaces of $T_z \in \mathcal{B}(L_a^2(D))$ has attracted a lot of attention from operator theorists and function theorists. Many results have been found(see [42] [43] [44]). Similarly a lot of work also has been given towards the description of the lattice $\text{Lat}(T_z) \cap \text{Lat}(T_w)$ for Toeplitz operators T_z and T_w on Hardy Space $H^2(T^2)$. (Consult [1] [18] [15] [36] [37] [50] [51] [67] [67] [10]) In [83], It is shown that the Bergman Shift is a super-isometrically dilatable operator with minimal super-isometrical dilation triple $\{T_z, T_w, \mathcal{H}^2(T^2)\}$. A natural question arises: what is the relationship between these two invariant space lattices. In this paper, we will study this relationship. We will derive an equivalent formulation for the invariant subspace problem, and we also prove a result about T_z, T_w similar to that of the Bergman shift (see [28]).

4.2 Some properties and an equivalent description of the invariant subspace lattice for the Bergman Shift

From [83], we have

$$L_a^2(D) \cong \mathcal{H}^2(T^2) \ominus \text{clos}\{(z-w)\mathcal{H}^2(T^2)\} = \mathcal{H},$$

where $\mathcal{H} = \text{span}\{p_n(z, w) : p_n = z^n + z^{n-1}w + \cdots + w^n, n \geq 0\}^-$. The Bergman shift T is the weighted shift $Te_n = \frac{\sqrt{n+1}}{\sqrt{n+2}}e_{n+1}$, on the space $\mathcal{H}$ with orthogonal base $\{e_n(z, w) = \frac{1}{\sqrt{n+1}}p_n(z, w)\}_{n=0}^{+\infty}$. We have

$$P_{\mathcal{H}}T_z^iT_w^j|_{\mathcal{H}} = T^{i+j},$$

and

$$T_z^*|_{\mathcal{H}} = T_w^*|_{\mathcal{H}} = T^*.$$

Here T_z and T_w are the Toeplitz operators on $\mathcal{H}^2(T^2)$ consisting of multiplication by z and w , respectively. $P_{\mathcal{H}}$ is the orthogonal projection from $\mathcal{H}^2(T^2)$ to $\mathcal{H}$. Let's define a map η from $\text{Lat}(T)$ into the set of all subspace of $\mathcal{H}^2(T^2)$ as follows:

$$\eta(\mathcal{M}) \equiv \mathcal{M} \oplus \text{clos}\{(z-w)\mathcal{H}^2(T^2)\} \quad \text{for any } \mathcal{M} \in \text{Lat}(T).$$

Now we have:

Proposition 4.1: *The map η defined as above is an one-one map from $\text{Lat}(T)$ into $\text{Lat}(T_z) \cap \text{Lat}(T_w)$.*

Proof: For any $\mathcal{M} \in \text{Lat}(T)$, and any $x \in \eta(\mathcal{M})$, since

$$\eta(\mathcal{M}) = \mathcal{M} \oplus \text{clos}\{(z-w)\mathcal{H}^2(T^2)\}$$

we have there exist $m \in \mathcal{M}$ and $h \in \text{clos}\{(z - w)\mathcal{H}^2(T^2)\}$ such that $x = m + h$. Since $P_{\mathcal{H}}T_z = T$ and

$$\begin{aligned}\mathcal{H}^2(T^2) &= \mathcal{H} \oplus \text{clos}\{(z - w)\mathcal{H}^2(T^2)\} \\ \text{clos}\{(z - w)\mathcal{H}^2(T^2)\} &\in \text{Lat}(T_z) \cap \text{Lat}(T_w)\end{aligned}$$

we have

$$T_z h \in \text{clos}\{(z - w)\mathcal{H}^2(T^2)\},$$

and

$$\begin{aligned}T_z m &= P_{\mathcal{H}}T_z m + (I - P_{\mathcal{H}})T_z m \in \mathcal{M} + (I - P_{\mathcal{H}})\mathcal{H}^2(T^2) \\ &= \mathcal{M} + \text{clos}\{(z - w)\mathcal{H}^2(T^2)\}.\end{aligned}$$

Hence $\eta(\mathcal{M}) \in \text{Lat}(T_z)$. A similar argument shows that $\eta(\mathcal{M}) \in \text{Lat}(T_w)$.

On the other hand, if $\mathcal{M} \oplus \text{clos}\{(z - w)\mathcal{H}^2(T^2)\} \in \text{Lat}(T_z) \cap \text{Lat}(T_w)$, for any $m \in \mathcal{M}$, $y \in \mathcal{H}^2(T^2) \ominus \mathcal{M} = (\mathcal{H} \ominus \mathcal{M}) \oplus \text{clos}\{(z - w)\mathcal{H}^2(T^2)\}$, we have $o \in \mathcal{H} \ominus \mathcal{M}$, $h \in \text{clos}\{(z - w)\mathcal{H}^2(T^2)\}$, such that $y = o + h$. We now compute:

$$\langle P_{\mathcal{H}}T_z m, y \rangle = \langle P_{\mathcal{H}}T_z m, o + h \rangle = \langle P_{\mathcal{H}}T_z m, o \rangle = \langle T_z m, o \rangle = 0.$$

Hence, $P_{\mathcal{H}}T_z \mathcal{M} \subset \mathcal{M}$; i.e., $T\mathcal{M} \subset \mathcal{M}$. Hence, η is a ont-to-one map.

By Theorem 1 in [73], we know that if $\mathcal{M}_1, \mathcal{M}_2 \in \text{Lat}(T)$, then there is a unitary operator V from $\mathcal{M}_1$ onto $\mathcal{M}_2$ such that $V(T|_{\mathcal{M}_1} = (T|_{\mathcal{M}_2})V$ if and only if $\mathcal{M}_1 = \mathcal{M}_2$. We have $\eta(\mathcal{M}_1) = \eta(\mathcal{M}_2)$; i.e., they are equivalent; i.e., there exists a unitary operator U from $\eta(\mathcal{M}_1)$ onto $\eta(\mathcal{M}_2)$ such that

$$T_z|_{\eta(\mathcal{M}_2)}U = UT_z|_{\eta(\mathcal{M}_1)} \quad \text{and}$$

$$T_w|_{\eta(\mathcal{M}_2)}U = UT_w|_{\eta(\mathcal{M}_1)}.$$

Before we continue, let's introduce the following definitions:

Definition 4.1: Let $\mathcal{S} \in \text{Lat}(T_z) \cap \text{Lat}(T_w) (\subseteq \mathcal{H}^2(T^2))$. We say $\mathcal{S}$ is of full range if

$$\text{span}\{(M_z^*)^j \mathcal{S}; j \geq 0\}^- = L^2(T, z) \otimes \mathcal{H}^2(T, z)$$

$$\text{span}\{(M_w^*)^j \mathcal{S}; j \geq 0\}^- = L^2(T, w) \otimes \mathcal{H}^2(T, w)$$

where M_z and M_w are multiplication operators on $L^2(T^2)$.

Definition 4.2: Let $T \in \mathcal{L}(\mathcal{H})$, we say $\text{Lat}(T)$ is saturated if for any $\mathcal{M}_1, \mathcal{M}_2 \in \text{Lat}(T)$ with $\mathcal{M}_1 \subset \mathcal{M}_2$ and $\dim(\mathcal{M}_2 \ominus \mathcal{M}_1) \geq 2$, there exists a invariant subspace $\mathcal{M} \in \text{Lat}(T)$ such that $\mathcal{M}_1 \subsetneq \mathcal{M} \subsetneq \mathcal{M}_2$.

Because of the universal dilation property of the Bergman Shift(see [14]), the invariant subspace problem is equivalent to the problem if Bergman Shift is saturated. We have the following result.

Lemma 4.1: If $\mathcal{U} = \mathcal{M} \oplus \text{clos}\{(z - w)\mathcal{H}^2(T^2)\} \in \text{Lat}(T_z) \cap \text{Lat}(T_w)$, then $\mathcal{U}$ is a full ranged subspace of $\mathcal{H}^2(T^2)$.

Proof: Because of the symmetry of z and w , we only need to prove

$$\text{span}\{(M_z^*)^j \mathcal{U}; j \geq 0\}^- = L^2(T, z) \otimes \mathcal{H}^2(T^2).$$

We need to prove

$$\text{span}\{(M_z^*)^j \{(z - w)\mathcal{H}^2(T^2)\}; j \geq 1\}^- = L^2(T, z) \otimes \mathcal{H}^2(T^2).$$

Since $(1 - \bar{z}w) = M_z^*(z - w)$ is an element of the set on the left of the last equality and since this space is invariant for M_z , M_z^* and M_w , the left side is invariant for $M_{\bar{z}w}$. Let $y = \bar{z}w$. Since $(1 - y)$ is an outer function for $\mathcal{H}^2(T, y)$ (see,[46, P76]) we have

$$\text{span}\{y^l(1 - y); l \geq 0\}^- = \mathcal{H}^2(T^2, y).$$

Since

$$\langle (\bar{z}w)^l, (\bar{z}w)^k \rangle_{L^2(T) \otimes L^2(T)} = \delta_{lk} = \langle y^l, y^k \rangle_{L^2(T)},$$

we have $1 \in \text{span}\{y^j(y-1); j \geq 0\}^- \subset L^2(T, y)$. Hence,

$$\begin{aligned} 1 &\in \mathcal{H}^2(T^2, y) \\ &\subset \text{span}\{(M_z^*)^j \{(z-w)\mathcal{H}^2(T^2)\}\}^- \end{aligned}$$

We see then that

$$\begin{aligned} &L^2(T, z) \otimes \mathcal{H}^2(T, w) \\ &= \text{span}\{M_z^j M_w^i \cdot 1; i \geq 0, j \text{ is any integer}\}^- \\ &\subset \text{span}\{(M_z^*)^j \{(z-w)\mathcal{H}^2(T^2)\}\}^-. \end{aligned}$$

We are done.

We say two subspaces $\mathcal{M}_i \in \text{Lat}(T)$ are equivalent if $T|_{\mathcal{M}_1}$ and $T|_{\mathcal{M}_2}$ are unitarily equivalent.

Proposition 4.2: *Assume $\mathcal{M}_1$ and $\mathcal{M}_2 \in \text{Lat}(T)$. The following are equivalent.*

- (a) $\mathcal{M}_1, \mathcal{M}_2$ are unitarily equivalent;
- (b) $\mathcal{M}_1 = \mathcal{M}_2$;
- (c) $\eta(\mathcal{M}_1)$ and $\eta(\mathcal{M}_2)$ are equivalent;
- (d) $\eta(\mathcal{M}_1) = \eta(\mathcal{M}_2)$

Proof: From Theorem 1 in [73], we have $(a) \iff (b)$. Now $(b) \implies (c)$, and $(d) \implies (b)$ follows from Proposition 1. Now $(c) \implies (d)$ follows from Corollary 2 in [1] and our lemma 4.1. We are done.

Corollary 4.1: *The invariant subspace question is the same as if $\{\mathcal{M} \in \text{Lat}(T_z) \cap \text{Lat}(T_w) : \text{clos}(z - w)\mathcal{H}^2(\mathbb{T}^2) \subset \mathcal{M}\}$ is saturated.*

The following will discuss some properties about these invariant subspaces. From Theorem 2 in [36] and Theorem in [67], we know that for any $\mathcal{M} \in \text{Lat}(T_z) \cap \text{Lat}(T_w)$ only one of the following is satisfied.

(a) There exists an inner function $f \in \mathcal{H}^2(\mathbb{T}^2)$ such that $\mathcal{M} = f\mathcal{H}^2(\mathbb{T}^2)$. Note $\mathcal{M}$ has this form

if and only if $T_z|_{\mathcal{M}}$ and $T_w|_{\mathcal{M}}$ are double commuting operators. We call this kind of space a H^2 - type space.

(b) $\mathcal{M}$ is an ultraevanescent type(see [36] for the definition) invariant subspace and $T_z|_{\mathcal{M}}$, $T_w|_{\mathcal{M}}$ are non-doubly commuting shifts on $\mathcal{M}$.

A natural question arises. Is the invariant subspace $\mathcal{M}_1 \oplus \text{clos}\{(z - w)\mathcal{H}^2(\mathbb{T}^2)\}$ is an H^2 -type space or an ultraevanecent type space? The following proposition will help us answer this question.

Proposition 4.3: *If $\mathcal{U} = \mathcal{M} \oplus \text{clos}\{(z - w)\mathcal{H}^2(\mathbb{T}^2)\} \in \text{Lat}(T_z) \cap \text{Lat}(T_w)$ then $\mathcal{U}$ is an H^2 - type subspace if and only if $\mathcal{U} = \mathcal{H}^2(\mathbb{T}^2)$.*

Proof: The sufficiency is obvious. Let's prove the necessity. If $\mathcal{U} \neq \mathcal{H}^2(\mathbb{T}^2)$ and is a H^2 - type space, then we have an inner function f such that $\mathcal{U} = f\mathcal{H}^2(\mathbb{T}^2)$. Now by Corollary 1 in [1] we know that $\mathcal{U}$ and $\mathcal{H}^2(\mathbb{T}^2)$ are unitarily equivalent. From our previous lemma and Corollary 2 in [1], we have $\mathcal{U} = \mathcal{H}^2(\mathbb{T}^2)$, a contradiction.

Now let's consider the other candidate for a universal dilatable operator which is also super-isomrtrically dilatable operator. To do this let's consider the restriction of a super-isometrically dilatable operator to one of its invariant subspaces.

Proposition 4.4: *If $T \in \mathcal{L}(\mathcal{H})$ is a super-isometrically dilatable operator with minimal dilation triple $\{U, V, \mathcal{K}\}$, then for any $\mathcal{M} \in \text{Lat}(T) \cap \text{Lat}(T^*)$, the operator $T|_{\mathcal{M}}$ is also super-isometrically dilatable, with the minimal super dilation triple $\{U|_{\mathcal{R}}, V|_{\mathcal{R}}, R\}$. Here $\mathcal{R} = \text{span}\{U^i V^j \mathcal{M}; i, j \geq 0\}^-$.*

Proof: Since T has the minimal dilation triple $\{U, V, \mathcal{K}\}$, then U and V are doubly commuting operators on $\mathcal{K}$. Since $P_{\mathcal{H}} U^i V^j|_{\mathcal{H}} = T^{i+j}$ and $\mathcal{M} \in \text{Lat}(T)$, we have

$$\begin{aligned} & P_{\mathcal{M}} U^i V^j|_{\mathcal{M}} \quad (P_{\mathcal{M}} \text{ is from } \mathcal{K} \text{ onto } \mathcal{M}) \\ &= P_{\mathcal{M}} P_{\mathcal{H}} U^i V^j|_{\mathcal{M}} \\ &= P_{\mathcal{M}} T^{i+j}|_{\mathcal{M}} \quad (P_{\mathcal{M}} \text{ is from } \mathcal{H} \text{ onto } \mathcal{M}) \\ &= (T|_{\mathcal{M}})^{i+j}. \end{aligned}$$

Now $U^*|_{\mathcal{H}} = V^*|_{\mathcal{H}} = T^*$ and since $\mathcal{M}$ is a reducing subspace of T , we have for any $x \in \mathcal{M}$ that $U^*x = T^*x = (T|_{\mathcal{M}})^*x$. Hence, $U^*|_{\mathcal{M}} = (T|_{\mathcal{M}})^*$. Similarly we have $V^*|_{\mathcal{M}} = (T|_{\mathcal{M}})^*$. It's not difficult to check the minimal Dilation space is $\mathcal{R}$ and the minimal Dilation triple is $\{U, V, \mathcal{R}\}$. We are done.

Now let's look at the Toeplitz operator T_{z^n} on Bergman space $L_a^2(D)$. From Theorem 2 in [59] or Theorem 3.3, we know its reducing subspace lattice has the form

$$\text{span}\{(T_{z^n})^l \{z^{i_1}, z^{i_2}, \dots, z^{i_k}\}; l \geq 0\}$$

with $0 \leq k \leq (n-1)$ and $0 \leq i_j \leq (n-1)$ for $j = 0, 1, \dots, k$. Let $\mathcal{M}_k = \text{span}\{T_{z^n}^l z^k; l \geq 0\}$.

It's easy to check that

$$T_{z^n}|_{\mathcal{M}_k} h_m(z) = \frac{\sqrt{mn+k+1}}{\sqrt{(m+1)n+k+1}} h_{m+1}(z)$$

where $\{h_m(z)\}_{m=0}^{+\infty} = \{\sqrt{mn+k+1}z^{mn+k}\}_{m=0}^{+\infty}$. From Chapter X. in [14] we know $M_{z^n}|_{\mathcal{M}_k}$ is also a universal dilatable operator, and by Theorem 1 in [83], for any inner function f with finite order, the operator T_f on Bergman Space is super-isometrically dilatable. Thus T_{z^n} is super isometrically dilatable. Using Proposition 4, we now have:

Proposition 4.5: *For any $n \geq 1$, and $1 \leq k \leq n$ we have that $T_{z^n}|_{\mathcal{M}_k}$ is super-isometrically dilatable weighted shift which also has the universal dilation property.*

4.3 On interpolating sequence on bidisk

A sequence $\{z_n\} \subset D$ is called an interpolating sequence for H^∞ if for each bounded sequence of complex numbers $\{w_n\}$ there exists $\phi \in H^\infty$ such that $\phi(z_n) = w_n$ for all n . Carleson [22] gave the following geometric characterization of interpolating sequences for H^∞ . For $z, w \in D$, let

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right|$$

be the pseudo-hyperbolic metric on D . If $I = \{e^{i\theta} : \theta_0 \leq \theta \leq \theta_0 + a\}$ is an arc on ∂D whose length = $|I| = a$, let

$$S(I) = \{re^{i\theta} : 1 - a \leq r < 1, \theta_0 \leq \theta \leq \theta_0 + a\}$$

be the approximate square with base I . If $a \geq 1$, we set $S(I) = D$.

Theorem[Carleson [22]]. *A sequence $\{z_n\} \subset D$ is an interpolating sequence for H^∞ if and only if there is an $\epsilon > 0$ and $K < \infty$ so that*

$$\rho(z_n, z_m) \geq \epsilon \text{ for all } n \neq m \text{ and}$$

$$\sum_{z_n \in S(I)} (1 - |z_n|^2) \leq K|I|$$

for all arcs $I \subset \partial D$.

Now let

$$A^{-n,2} = \{f \text{ analytic in } D : \int_D |f(z)|^2 (1 - |z|^2)^{2n-1} dA(z) < \infty\}$$

for each $n > 0$. The sequence $\{z_n\}$ is a set of interpolation for $A^{-n,2}$ if for every sequence $\{a_n\}$ for which $\sum (1 - |z_n|^2)^{2n+1} |a_n|^2 < \infty$ there exists a function $f \in A^{-n,2}$ such that $f(z_n) = a_n$ for all n . We say that a sequence $\Gamma = \{z_n\}$ is uniformly discrete if

$$\inf_{j \neq k} \rho(z_j, z_k) > 0$$

For a uniformly discrete sequence $\Gamma = \{z_n\}$ and number $\frac{1}{2} < r < 1$ let

$$D(\Gamma, r) = \frac{\sum_{\frac{1}{2} < |z_n| < r} \log \frac{1}{|z_n|}}{\log \frac{1}{1-r}}.$$

For each $z \in D$ we define a new sequence

$$\Gamma_z = \left\{ \frac{z_n - z}{1 - \bar{z}z_n} \right\}.$$

We call

$$D^-(\Gamma) = \lim_{r \rightarrow 1} \inf_{z \in D} D(\Gamma_z, r)$$

and

$$D^+(\Gamma) = \overline{\lim}_{r \rightarrow 1} \sup_{z \in D} D(\Gamma_z, r),$$

the lower and upper uniform densities of Γ , respectively. In [77] Seip obtained the following result.

Theorem[Seip] *A sequence Γ of distinct points in D is an interpolating sequence for $A^{-n,2}$ if and only if Γ is uniformly discrete and $D^+(\Gamma) < n$.*

When $n = \frac{1}{2}$, Seip's theorem will be a special case for our standard Bergman space result. A natural question arises: What are the interpolating sequences for the Hardy space on the bidisk? Little is known. In the following part of this section, we will use Seip's theorem and the super-isometrically dilatability of the Bergman shift to study a special case of this question. From our result, we will see the characterization of an interpolating sequence in the Bergman space is just a special case of the Bidisk case. We will see that the corresponding question about Bidisk is much more difficult. The following theorem coming from [83] is essentially due to R.G.Douglas and V.I.Paulsen (Lemma 5.5 of [29]).

Theorem 4.0: *Recall for each integer n , that*

$$p_n(z, w) = z^n + z^{n-1}w + \cdots + w^n,$$

and T_z, T_w stand for the analytic Toeplitz operator, multiplying by z and w , respectively, on $\mathcal{H}^2(T^2)$. If

$$\mathcal{H} = \text{span}\{p_n(z, w), n = 0, 1, 2, \dots\}^- (\subset \mathcal{H}^2(T^2)).$$

Then

$$\mathcal{H}^2(T^2) = \mathcal{H} \oplus \text{clos}\{(z - w)\mathcal{H}^2(T^2)\}.$$

Moreover, $\{T_z, T_w, \mathcal{H}^2(T^2)\}$ is the minimal super-isometric dilation of the Bergman shift M .

Proof: We prove the decomposition about $\mathcal{H}^2(T^2)$ first. Assume that $f(z, w) \in \mathcal{H}^2(T^2)$, $f(z, w) = \sum_{i,j=0}^{+\infty} a_{ij} z^i w^j = \sum_{n=0}^{+\infty} Q_n(z, w)$, where $Q_n(z, w)$ is a homogeneous polynomial of

degree n . Obviously,

$$Q_k(z, w) \perp Q_l(z, w), \text{ for } k \neq l.$$

If $Q_n(z, w) \perp (z - w)\mathcal{H}^2(T^2)$, we have

$$\langle Q_n, (z - w)\mathcal{H}^2(T^2) \rangle = \langle (T_z^* - T_w^*)Q_n, \mathcal{H}^2(T^2) \rangle = 0.$$

That is,

$$\begin{aligned} T_z^* Q_n(z, w) &= T_w^* Q_n(z, w) \\ T_z^* \left(\sum_{i+j=n} a_{ij} z^i w^j \right) &= T_w^* \left(\sum_{i+j=n} a_{ij} z^i w^j \right), \end{aligned}$$

or equivalently,

$$\begin{aligned} \sum_{i=1}^n a_{i, n-i} z^{i-1} w^{n-i} &= \sum_{i=0}^n a_{i, n-i} z^i w^{n-i-1} \\ \sum_{i=0}^{n-1} a_{i+1, n-i-1} z^i w^{n-i-1} &= \sum_{i=0}^{n-1} a_{i, n-i} z^i w^{n-i-1}. \end{aligned}$$

Hence, we have

$$a_{i+1, n-i-1} = a_{i, n-i} \quad i = 0, 1, 2, \dots, n-1.$$

Thus,

$$a_{0, n} = a_{1, n-1} = a_{2, n-2} = \dots = a_{n, 0} \equiv a_n$$

and

$$Q_n(z, w) = a_n p_n(z, w).$$

Therefore, if $f(z, w) \perp (z - w)\mathcal{H}^2(T^2)$, then we conclude

$$f(z, w) = \sum_{n=0}^{\infty} a_n p_n \in \mathcal{H}.$$

This yields the asserted decomposition about $\mathcal{H}^2(T^2)$.

Now consider the operator $P_{\mathcal{H}}T_z|_{\mathcal{H}}$. We have,

$$\begin{aligned} P_{\mathcal{H}}T_z 1 &= P_{\mathcal{H}} \left(\frac{z+w}{2} + \frac{z-w}{2} \right) \\ &= \frac{z+w}{2} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \frac{z+w}{2} \\ &= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} p_1. \end{aligned}$$

In general, it is easy to see by this computation that

$$\begin{aligned} P_{\mathcal{H}}T_z \frac{1}{\sqrt{n+1}} p_n &= P_{\mathcal{H}} \left(\frac{\sqrt{n+1}}{\sqrt{n+2}} \frac{1}{\sqrt{n+2}} p_{n+1} \right. \\ &\quad \left. + \frac{z^{n+1} - w^{n+1} + z^n w - w^{n+1} + \cdots + z w^n - w^{n+1}}{(n+2)\sqrt{n+1}} \right) \\ &= \frac{\sqrt{n+1}}{\sqrt{n+2}} \frac{p_{n+1}}{\sqrt{n+2}}. \end{aligned}$$

Hence, $P_{\mathcal{H}}T_z$ is actually the Bergman shift M on $\mathcal{H}$ with respect to the base

$$\left\{ \frac{1}{\sqrt{n+1}} p_n, n = 0, 1, 2, \dots \right\}.$$

Similarly, $P_{\mathcal{H}}T_w|_{\mathcal{H}} = M$. At the same time, we find that

$$T_z T_w^* z^i w^j = T_z z^i w^{j-1} = z^{i+1} w^{j-1} = T_w^* T_z z^i w^j, \quad \forall i \geq 0, j \geq 0,$$

$$T_z T_w^* z^i = T_w^* T_z z^i = 0, \quad \text{for all } i \geq 0.$$

It follows that T_z, T_w are doubly commuting isometries on $\mathcal{H}^2(T^2)$. It is easy to see that

$$\ker T_z^* \cap \ker T_w^* = \{\alpha 1 : \alpha \in \mathbb{C}\}.$$

Moreover, we have

$$T_z^* \frac{1}{\sqrt{n+1}} p_n = T_w^* \frac{1}{\sqrt{n+1}} p_n = \frac{n}{\sqrt{n+1}} \frac{1}{\sqrt{n}} p_{n-1},$$

and consequently

$$T_z^*|_{\mathcal{H}} = T_w^*|_{\mathcal{H}} = M^*.$$

Noting that $\text{clos}\{(z - w)\mathcal{H}^2(T^2)\}$ is invariant for T_z and T_w , we can check that

$$P_{\mathcal{H}}T_z^iT_w^j|_{\mathcal{H}} = M^{i+j}.$$

Since $\mathcal{H}^2(T^2) = \text{span}\left\{\sum_{i,j \geq 0} a_{ij} z^i w^j\right\} = \text{span}\left\{\sum_{i,j \geq 0} z^i w^j \mathcal{H}\right\}$, by Proposition 1 in [83], we conclude that $\{T_z, T_w, \mathcal{H}^2(T^2)\}$ is the minimal super-isometric dilation of the Bergman shift M .

We say a sequence $\{(z_n, w_n)\}$ in D^2 an interpolating sequence for Hardy space $\mathcal{H}^2(T^2)$ if for every sequence $\{a_n\}$ for which $\sum(1 - |z_n w_n|)^2 |a_n|^2 < \infty$, there exists a function $f \in \mathcal{H}^2(T^2)$ such that $f(z_n, w_n) = a_n$ for all n . Now we have

Theorem 4.01: *For any sequence $\{z_n\}$ in D , the sequence $\{(z_n, z_n)\}$ is an interpolating sequence for Hardy space $\mathcal{H}^2(T^2)$ if and only if $\{z_n\}$ is an interpolating sequence for the Bergman space $L_a^2(D)$.*

Proof: From Theorem 4.0, we see that

$$L_a^2(D) \cong H^2(T^2) \ominus \text{clos}\{(z - w)H^2(T^2)\} \equiv \mathcal{H}.$$

Now let's define a map U from $L_a^2(D)$ to $\mathcal{H}$ by setting

$$U\left(\sum_{n=0}^{+\infty} a_n \sqrt{n+1} z^n\right) \equiv \sum_{n=0}^{+\infty} a_n \frac{1}{\sqrt{n+1}} p_n$$

for any sequence $\{a_n\}$ for which

$$\sum_{n=0}^{+\infty} |a_n|^2 < +\infty.$$

Since $\{\sqrt{n+1} z^n\}$ is an orthogonal base for $L_a^2(D)$, and $\left\{\frac{1}{\sqrt{n+1}} p_n\right\}$ is an orthogonal base for $\mathcal{H}$, it's obvious that the map U is a unitary operator between these two spaces. If

$\{z_n\}$ is an interpolating sequence for $L_a^2(D)$, it implies for any sequence $\{a_n\}$ for which $\sum(1 - |z_n|^2)^2|a_n|^2 < \infty$ there exists a function

$$f(z) = \sum_{j=0}^{+\infty} f_j \sqrt{j+1} z^j$$

such that $f(z_n) = a_n$. Hence, we can choose function

$$g(z, w) \equiv (Uf)(z, w) = \sum_{j=0}^{+\infty} f_j \frac{1}{\sqrt{j+1}} p_j$$

in $\mathcal{H}$ such that

$$\begin{aligned} g(z_n, z_n) &= \sum_{j=0}^{+\infty} f_j \frac{1}{\sqrt{j+1}} (z_n^j + z_n^{j-1} z_n + \cdots + z_n^j) \\ &= \sum_{j=0}^{+\infty} f_j \frac{1}{\sqrt{j+1}} (j+1) z_n^j \\ &= \sum_{j=0}^{+\infty} f_j \sqrt{j+1} z_n^j \\ &= f(z_n) = a_n. \end{aligned}$$

Using the inverse of U , we can prove the other direction of this theorem using a similar argument.

Now combining the theorem 4.01 and Seip's theorem, we have

Theorem 4.02: *For a sequence $\Gamma = \{z_n\}$ of distinct points i in D , the sequence $\{(z_n, z_n)\}$ is an interpolating sequence for $H^2(T^2)$ if and only if Γ is uniformly discrete and $D^+(\Gamma) < \frac{1}{2}$.*

Remark: As in the Bergman space setting, we can easily prove any interpolating sequence for $H^2(T^2)$ is a zero set. Note the characterization of a zero set is also much more complicated than Bergman space. One of the well known results in this area is that a zero set is submanifold of C^2 .

4.4 On equivalent class

In this section, we will examine a special case in which both the canonical model and the minimal super dilation model are unique. The only restriction about this special case is the commutant $\{U, V, U^*, V^*\}'$ in $\mathcal{L}(\mathcal{K})$ is trivial. A corollary of this work shows a new property of the Bergman Shift. In Hardy space a similar property has been proved in [28]. Let $\mathcal{M}$ be an invariant subspace of $T \in \mathcal{L}(\mathcal{H})$ and set $\mathcal{O}$ to be the complement $\mathcal{H} \ominus \mathcal{M}$, and set $S = P_{\mathcal{O}}T|_{\mathcal{O}}$. The super dilation of T is also an isometric dilation of S Hence it also provide a canonical model for S . We will show it is also unique. In fact the super dilation itself is also unique for S . We will use the basic results about super isometrically dilatable operators in [83] and Theorem 2 in [28] to prove our conclusion. The following is our main result.

Theorem 4.1: *Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are two invariant subspaces for T , a super isometrically dilatable operator on Hilbert space $\mathcal{H}$. Also suppose the minimal super dilated triple is $\{U, V, \mathcal{K}\}$ with the commutant $\{U, V, U^*, V^*\}'$ trivial. There exists a unitary operator $\mathbf{V} : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ satisfying $\mathbf{V}S_1 = S_2\mathbf{V}$ if and only if $\mathcal{M}_1 = \mathcal{M}_2$ if and only if $\mathcal{O}_1 = \mathcal{O}_2$.*

Proof. From Proposition 2 in [14], we know

$$\mathcal{K} = \mathcal{H} \oplus \{h : U^*h = V^*h \text{ and } h \perp \mathcal{H}\} \oplus \text{clos}\{(U - V)\mathcal{K}\}.$$

From the definition of a super isometrically dilatable operator we have

$$P_{\mathcal{H}}U^iV^j|_{\mathcal{H}} = T^{i+j},$$

where $P_{\mathcal{H}}$ is orthogonal projection from $\mathcal{K}$ to $\mathcal{H}$. Hence, we have

$$P_{\mathcal{O}_i}T|_{\mathcal{O}_i} = P_{\mathcal{O}_i}(P_{\mathcal{H}}U|_{\mathcal{H}})|_{\mathcal{O}_i} \text{ and}$$

$$P_{\mathcal{O}_i}T|_{\mathcal{O}_i} = P_{\mathcal{O}_i}(P_{\mathcal{H}}V|_{\mathcal{H}})|_{\mathcal{O}_i},$$

($i = 1, 2$), where $P_{\mathcal{O}_i}$ is orthogonal projection from $\mathcal{H}$ to $\mathcal{O}_i$. Since $\mathcal{O}_i$ is a subspace of $\mathcal{H}$ ($i = 1, 2$), we have

$$P_{\mathcal{O}_i}T|_{\mathcal{O}_i} = P_{\mathcal{O}_i}U|_{\mathcal{O}_i}$$

and

$$P_{\mathcal{O}_i}T|_{\mathcal{O}_i} = P_{\mathcal{O}_i}V|_{\mathcal{O}_i}.$$

Now the projection on the right hand side is from $\mathcal{K}$ to $\mathcal{O}_i$, hence, we have

$$\mathbf{V}(P_{\mathcal{O}_1}U|_{\mathcal{O}_1}) = (P_{\mathcal{O}_2}U|_{\mathcal{O}_2})\mathbf{V}$$

and

$$\mathbf{V}(P_{\mathcal{O}_1}V|_{\mathcal{O}_1}) = (P_{\mathcal{O}_2}V|_{\mathcal{O}_2})\mathbf{V}.$$

On the other hand,

$$\mathcal{K} = \mathcal{M}_1 \oplus \{h : U^*h = V^*h \text{ and } h \perp \mathcal{H}\} \oplus \text{clos}\{(U - V)\mathcal{K}\} \oplus \mathcal{O}_1, \text{ and}$$

$$\mathcal{K} = \mathcal{M}_2 \oplus \{h : U^*h = V^*h \text{ and } h \perp \mathcal{H}\} \oplus \text{clos}\{(U - V)\mathcal{K}\} \oplus \mathcal{O}_2.$$

Now we will prove $\mathcal{K} \ominus \mathcal{O}_i$ are invariant subspaces for both U and V , ($i = 1, 2$). For any $x \in \mathcal{K} \ominus \mathcal{O}_1$ Let $x = m \oplus h \oplus k$, where $m \in \mathcal{M}_1, k \in \text{clos}\{(U - V)\mathcal{K}\}$, and $h \in \{h : U^*h = V^*h, h \perp \mathcal{H}\}$.

We will prove $Ux \in \mathcal{K} \ominus \mathcal{O}_1$, to do this, let's prove $Uk \in \text{clos}\{(U - V)\mathcal{K}\} \oplus \mathcal{O}_1$, $Uh \in \mathcal{K} \ominus \mathcal{O}_1$ and $Um \in \mathcal{K} \ominus \mathcal{O}_1$ respectively. It's easy to see

$$Uk \in \text{clos}\{(U - V)\mathcal{K}\} \subset \mathcal{K} \ominus \mathcal{O}_1.$$

Now for any $y \in \mathcal{H}$, by the definition of a super isometrically dilatable operator, we have

$$U^*|_{\mathcal{H}} = V^*|_{\mathcal{H}} = T^*.$$

Hence

$$U^*y \in \mathcal{H}$$

and

$$\langle Uh, y \rangle = \langle h, U^*y \rangle = 0.$$

This implies

$$Uh \in \mathcal{K} \ominus \mathcal{H} \subset \mathcal{K} \ominus \mathcal{O}_1.$$

For Um , since

$$P_{\mathcal{H}}U|_{\mathcal{H}} = T$$

and $\mathcal{M}_1 \subset \mathcal{H}$ and $\mathcal{M}_1 \in \text{Lat}(T)$, suppose

$$Um = m' \oplus o \oplus h' \oplus k'$$

with $o \in \mathcal{O}_1$, we have

$$Tm = P_{\mathcal{H}}Um = m' \oplus o \in \mathcal{M}_1,$$

hence, $o = 0$ i.e., $Um \in \mathcal{K} \ominus \mathcal{O}_1$. The same argument for V , and for $\mathcal{K} \ominus \mathcal{O}_2$, do the same thing. Now everything in Theorem 2 of [28] is satisfied. We are done.

As an application of the above theorem, we now consider the Bergman shift. Let $L_a^2(D)$ be the Bergman space and let $H^2(T^2)$ be the Hardy space on the polydisk D^2 , and let $P_n(z, w) = z^n + z^{n-1}w + \cdots + w^n$, T_z and T_w stand for the analytic Toeplitz operators, multiplying by z and w respectively on $H^2(T^2)$. Let $\mathcal{H} = \text{span}\{P_n(z, w), n = 0, 1, 2, 3, \dots\}^- (\subset H^2(T^2))$. By Theorem 2 in [14], we have

$$L_a^2(D) \cong H^2(T^2) \ominus \text{clos}\{(z - w)H^2(T^2)\} = \mathcal{H},$$

and the triple $\{T_z, T_w, H^2(T^2)\}$ is the minimal super isometric dilation of the Bergman shift. It is not difficult to check this dilation satisfies the hypothesis of the last Theorem. Hence, we have the following result.

Corollary 4.1: *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be invariant subspaces for the Bergman shift on Bergman Space $L_a^2(\mathbb{D})$, Let $\mathcal{O}_i = L_a^2(\mathbb{D}) \ominus \mathcal{M}_i$, and $P_{\mathcal{O}_i}$ be the projections from $L_a^2(\mathbb{D})$ to $\mathcal{O}_i$, then there exists unitary operator $\mathbf{V}$ from $\mathcal{O}_1$ to $\mathcal{O}_2$ such that $(P_{\mathcal{O}_2}\mathcal{M}_z|_{\mathcal{O}_2})\mathbf{V} = \mathbf{V}(P_{\mathcal{O}_1}\mathcal{M}_z|_{\mathcal{O}_1})$ if and only if $\mathcal{O}_1 = \mathcal{O}_2$ if and only if $\mathcal{M}_1 = \mathcal{M}_2$.*

Remark. Douglas and Foias's proof for the corresponding Hardy space setting depend on the properties of the appropriate isometry. This can't be used here, as a super isometrically dilatable operator generally is not an isometry. For example, the Bergman shift. Even it is an isometry it may not be super isometrically dilatable, for example the unilateral shift.

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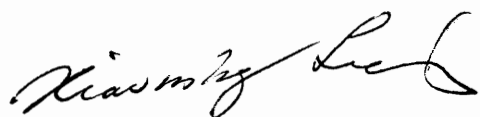
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A handwritten signature in black ink, appearing to read 'Xiaomeng Lu'. The signature is fluid and cursive, with the first name 'Xiaomeng' written in a more compact, stylized manner and the last name 'Lu' written in a larger, more open script.